Foundations for General Prospect Theory Through Probability Midpoint Consistency

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Abstract: A behavioural condition is proposed – probability midpoint consistency – which, in the presence of standard preference conditions, delivers prospect theory for arbitrary outcomes. A feature of our derivation is that a priori the reference point does not need to be known. Probability midpoint consistency entails a test for reference dependence that allows for the identification of the reference point.

Keywords: probability midpoint, prospect theory, rank dependence, reference dependence, sign dependence.

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1 Introduction

Kahneman and Tversky (1979) provided us with a powerful descriptive theory for decision under risk that integrates behavioral findings from psychology into economics. The three major advancements of original prospect theory concern reference dependence (outcomes are gains or losses relative to a reference point), loss aversion (a loss leads to greater disutility than the utility of a comparative gain) and sign dependence (decision weights for gains differ from those for losses). Later, in Tversky and Kahneman (1992), prospect theory[2] (PT) was extended to uncertainty and ambiguity by incorporating the requirements of rank-dependence introduced by Quiggin (1981, 1982) for risk and by Schmeidler (1989) for ambiguity, and it received a sound preference foundation by using the tools underlying continuous utility measurement developed in Wakker (1989); see also Wakker and Tversky (1993). Due to its ability to incorporate and account for sign-dependent probabilistic risk attitudes, ambiguity attitudes, reference-dependence, loss aversion and diminishing sensitivity in outcomes and probabilities, PT has become one of the most well-known descriptive theories for risk and uncertainty (Starmer 2000, Kahneman and Tversky 2000, Wakker 2010).

The aim of this paper is to provide a behavioral preference foundation for PT for decision under risk without assuming prior knowledge of a reference point. To position our contribution it is important to briefly recall the existing PT-foundations for risk. Remarkably, it took many years since the 1979' model to develop the first foundations of PT for decision under risk; this was done by Chateauneuf and Wakker (1999). Later, Kothiyal, Spinu and Wakker (2011) provided foundations of PT for continuous probability distributions. More recently, Schmidt and Zank (2012) derived PT with endogenous reference points by exploiting sign dependence and diminishing sensitivity of the utility. All these theoretical developments assumed that the set of outcomes is endowed with a sufficiently rich structure that allows for the derivation of continuous cardinal utility.

[2] Some authors prefer to distinguish the original prospect theory of Kahneman and Tversky (1979) from the modern version, cumulative prospect theory, of Tversky and Kahneman (1992). Indeed, as Wakker (2010, Apendix 9.8) clarifies, in general these models make different predictions. Here we restrict attention to the modern version, and hence, we use the shorter name *prospect theory*.

This paper takes a different approach to obtain foundations for PT. It does not assume richness of the set of outcomes but, instead, it follows the traditional approach pioneered by von Neumann and Morgenstern (1944) of using the natural structure given by the probability interval. This approach has been used to derive preference foundations for rank-dependent utility (RDU) by Chateauneuf (1999), Abdellaoui (2002) and Zank (2010); specific parametric probability weighting functions were provided by Diecidue, Schmidt and Zank (2009), Abdellaoui, l'Haridon and Zank (2010) and Webb and Zank (2011). Neither of these results have looked at PT-preferences,[3] although, intuitively, most of those preference foundations for RDU can be extended to PT if the reference point is given. In the absence of this information such extensions become a challenge. This may explain why until now PT has not been derived using the "probabilistic approach." We fill this gap and show that PT can be obtained from preference conditions where objective probabilities are given and the set of outcomes can be very general. The reference point in our approach is revealed through probabilistic risk behavior and its existence is not assumed a priori. This shows that, also in this respect, our model extends all existing PT-foundations for risk. There is a related literature discussing reference point formation in dynamic settings (Shalev 2000, 2002, Rabin and Kőszegi 2006, Kőszegi 2010). Our work is complementary as it provides existence results for reference points in the traditional static framework.

The importance of having sound preference foundations for decision models, in particular for PT, has recently been reiterated by Kothiyal, Spinu and Wakker (2011, pp. 196–197). If a continuous utility is not available, as a result of outcomes being discrete (e.g., as in health or insurance), the relationship between the empirical primitive (i.e., the preference relation) and the assumption of PT becomes unclear, which is undesirable. In that case one can no longer be sure that the predictions and estimates are in line with the behavior underlying the preferences. The conditions presented here are necessary and sufficient for PT and, therefore, they help to clarify which assumptions one makes by invoking the model. In particular, the new foundations highlight the difference between expected

[3] An exception is Prelec (1998), where PT is assumed, however, the key preference condition there requires a continuous utility.

utility, RDU and PT in a transparent way.

Our key preference condition is based on the idea of probability midpoint elicitations. If we know the reference point, our condition simply requires that elicited probability midpoints are independent of the outcomes (i.e., the stimuli) used to derive those midpoints, whenever all outcomes are of the same sign (i.e., either all outcomes are gains or all are losses). Indeed, under PT the probability weighting function for probabilities of gains may be different to the probability weighting function for probabilities of losses. This feature, called *sign-dependence*, has widely been documented (Edwards 1953, 1954, Hogarth and Einhorn 1990, Tversky and Kahneman 1992, Abdellaoui 2000, Bleichrodt, Pinto, and Wakker 2001, Etchard-Vincent 2004, Payne 2005, Abdellaoui, Vossmann and Weber 2005, Abdellaoui, l'Haridon and Zank 2010).[4] The original elicitation technique for nonparametric probability weighting functions was presented by Abdellaoui (2000) and Bleichrodt and Pinto (2000). They invoke utility measurements prior to the elicitation of probability weighting functions. A simplified version of this method appeared recently in van de Kuilen and Wakker (2011) and requires a single utility midpoint elicitation. In contrast, the method of Wu, Wang and Abdellaoui (2005) can be applied in the probability triangle and does not necessitate utility midpoint elicitation. We assume probabilities are given and extend these methods to derive PT axiomatically. In this way we obtain preference conditions that are empirically meaningful jointly with the provision of behavioral foundations of PT for decision under risk.

Our elicitation tool for probability midpoints is based on joint shifts in probabilities away from intermediate outcomes. Additionally, and central to our preference foundation for PT, is the incorporation of a behavioral test for sign-dependence. We invoke consistency of probability shifts to worse outcomes and consistency of probability shifts to better outcomes for given pairs of prospects. If no sign-dependence is revealed, the two midpoint consistency requirements become compatible and, thus,

[4] Sign-dependence is one of the consequences of reference dependence. The latter serves as the key explanation for prominent phenomena like the disparity between willingness to pay and willingness to accept (Kahneman, Knetsch, and Thaler 1990, Bateman et al. 1997, Viscusi, Magat and Huber 1987, Viscusi and Huber 2012), the endowment effect (Thaler 1980, Loewenstein and Adler 1995), and the status quo bias (Samuelson and Zeckhauser, 1988).

they imply RDU. In that case it is impossible to obtain a distinction of outcomes into gains and losses by looking at probabilistic risk attitudes. However, if sign-dependence is present, an incompatibility of the two consistency properties is revealed. As a result outcomes can be divided into two disjoint sets with consistency of probability midpoints holding on each set. That is, there must be a special outcome, i.e., the reference point, that demarcates the set of gains from the set of losses. In the presence of standard preference conditions, the two principles of consistency are sufficient to obtain either RDU (i.e., absence of reference points), a special case of PT, or genuine PT with sign-dependent probability weighting. In addition, some extreme forms of optimistic and pessimistic behavior are permitted. Although these are compatible with PT, the corresponding preference functionals are more general. Specifically, if consistency reveals that there is only one gain, it may not be possible to separate the weighting function for gain probabilities from the utility of that gain due to asymptotic behavior for probabilities close to one. Similarly, consistency may reveal that there is only one loss and the weighting function for loss probabilities is unbounded at 1. As our objective is to avoid any structural assumptions on the set of outcomes we cannot exclude these extreme cases. The derived class of preference functionals can be combined as *general prospect theory.*

Next we present preliminary notation and recall the standard preference condition with implications thereof. In Section 3 we elaborate and present our main preference condition and the theorem. Extensions are discussed in Section 4 and the concluding remarks in Section 5 are followed by an appendix with proofs.

2 Preliminaries

In this section we recall the standard ingredients for decision under risk and the traditional preference conditions that are shared by expected utility and prospect theory.

2.1 Notation

Let X denote the set of *outcomes*. Initially, we make several simplifying assumptions. In Section 4, these are relaxed to demonstrate the full generality of our approach. First, we assume a finite set of outcomes, such that $X = \{x_1, \ldots, x_n\}$, with $n \geq 4$. A *prospect* is a finite probability distribution over X. Prospects can be represented by $P = (p_1, x_1; \ldots; p_n, x_n)$ meaning that outcome $x_j \in X$ is obtained with probability p_j, for $j = 1, \ldots, n$. Naturally, $p_j \geq 0$ for each $j = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$. Let $\mathcal{L}$ denote the set of all prospects.

A *preference relation* $\succcurlyeq$ is assumed over $\mathcal{L}$, and its restriction to subsets of $\mathcal{L}$ (e.g., all degenerate prospects where one of the outcomes is received for sure) is also denoted by $\succcurlyeq$. The symbols $\succ$ (strict preference) and $\sim$ (indifference) are defined as usual. We assume that no two outcomes in X are indifferent; they are ordered from best to worst, i.e., $x_1 \succ \cdots \succ x_n$. As in this section the outcomes are fixed we drop them from the notation without loss of generality.

Recall, that under *expected utility* (EU) prospects are evaluated by

$$EU(p_1, \ldots, p_n) = \sum_{j=1}^{n} p_j u(x_j), \tag{1}$$

with a *utility* function, u, which assigns to each outcome a real number and is strictly monotone (that is, u agrees with the preference ordering over outcomes: $u(x_i) \geq u(x_j) \Leftrightarrow x_i \succcurlyeq x_j$, $i, j \in \{1, \ldots, n\}$). Under EU the utility is *cardinal*, i.e., it is unique up to multiplication by a positive constant and translation by a location parameter.

A more general model is *rank-dependent utility* (RDU) where prospect $P = (p_1, \ldots, p_n)$ is evaluated by[5]

$$RDU(p_1, \ldots, p_n) = \sum_{j=1}^{n} [w(p_1 + \cdots + p_j) - w(p_1 + \cdots + p_{j-1})] u(x_j). \tag{2}$$

Utility is similar to EU, however, RDU involves a weighting function for probabilities, w, that is

[5] As usual, we use the convention that the sum $\sum_{j=i}^{m} f_j = 0$ when $m < i$.

uniquely determined. Formally, the *weighting function*, w, is a mapping from the probability interval $[0, 1]$ into $[0, 1]$ that is strictly increasing with $w(0) = 0$ and $w(1) = 1$. In this paper the axiomatically derived weighting functions are continuous on $[0, 1]$. There is, however, empirical and theoretical interest in discontinuous weighting functions at 0 and at 1 (Kahneman and Tversky 1979, Birnbaum and Stegner 1981, Bell 1985, Cohen 1992, Wakker 1994, 2001, Chateauneuf, Eichberger and Grant 2007, Webb and Zank 2011, Andreoni and Sprenger 2009, 2012). We discuss relaxing the continuity assumption at the extreme probabilities in Section 4. It is well known that RDU reduces to EU if w is linear.

The main model of interest in this paper extends RDU by incorporating reference-dependence: the model assumes an outcome $x_k \in X$, $1 \leq k \leq n$, exists, such that outcomes preferred to it are *gains* and outcomes worse than it are *losses*. This may have the implication that, in contrast to RDU, the weighting function will depend on whether the weighted (decumulative) probabilities are those of gains or of losses. For this reason the term *sign-dependence* is used to highlight that the nonlinear treatment of decumulative probabilities depends on the sign of the outcome attached to each probability. Under *Prospect Theory* (PT) prospect $P = (p_1, \ldots, p_n)$ is evaluated by

$$\begin{aligned} PT(p_1, \ldots, p_n) = \; & \textstyle\sum_{j=1}^{k-1} [w^+(p_1 + \cdots + p_j) - w^+(p_1 + \cdots + p_{j-1})] u(x_j) \\ & + \textstyle\sum_{j=k}^{n} [w^-(p_j + \cdots + p_n) - w^-(p_{j+1} + \cdots + p_n)] u(x_j), \end{aligned} \tag{3}$$

where $u(x_k) = 0$; w^+ and w^- are continuous and strictly increasing probability weighting functions for decumulative probabilities of gains and losses, respectively. Under PT the utility is a *ratio scale* (i.e., it is unique up to multiplication by a positive constant) and the weighting functions are uniquely determined. If the dual probability weighting function for losses, $\hat{w}^-(p) := 1 - w^-(1 - p)$, for all $p \in [0, 1]$, is identical to w^+, then PT reduces to RDU. In that case we do not have sign-dependence.

As mentioned in the introduction, several preference foundations for PT have been proposed using the approach based on continuous utility. Foundations with general continuous utility include Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Köbberling and Wakker (2003, 2004), Wakker (2010), Kothiyal, Spinu and Wakker (2011), and Schmidt and Zank (2012). Derivations of CPT with specific forms of the utility function (linear/exponential, power, and variants of multiattribute utility) have been provided in Zank (2001), Wakker and Zank (2002), Schmidt and Zank (2009). Bleichrodt, Schmidt and Zank (2009) assume attribute specific reference points for derivations of functionals that combine PT and multiattribute utility. In the next subsection we present the standard preference conditions that all functionals presented in this section have to satisfy.

2.2 Traditional Preference Conditions

This subsection presents the classical preference conditions that are necessary for EU, RDU and PT. We are interested in conditions for a preference relation, $\succcurlyeq$, on the set of prospects $\mathcal{L}$ that *represent* $\succcurlyeq$ by a function, V, that assigns a real value to each prospect, such that for all $P, Q \in \mathcal{L}$,

$$P \succcurlyeq Q \Leftrightarrow V(P) \geq V(Q).$$

A requirement for the representation is that $\succcurlyeq$ is a *weak order*, i.e., the following axiom holds:

Weak Order: The preference relation $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all $P, Q \in \mathcal{L}$) and transitive.

Further requirements are those of first order stochastic dominance and of continuity in probabilities.

Dominance: The preference relation satisfies *first order stochastic dominance* (or *monotonicity* in decumulative probabilities) if $P \succ Q$ whenever $\sum_{j=1}^{i} p_j \geq \sum_{j=1}^{i} q_j$ for all $i = 1, \ldots, n$ and $P \neq Q$.

Continuity: The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of prospects $\mathcal{L}$ if for all prospects $P \succ Q$ and R there exist $\rho, \mu \in (0, 1)$ such that $\rho P + (1 - \rho)R \succ Q$ and

$P \succ \mu R + (1-\mu)Q$.[6]

A monotonic weak order that satisfies Jensen-continuity on $\mathcal{L}$ also satisfies the stronger Euclidean-continuity on $\mathcal{L}$ (see, e.g., Abdellaoui 2002, Lemma 18). Further, the three conditions taken together imply the existence of a continuous function $V : \mathcal{L} \to \mathbb{R}$, strictly increasing in each decumulative probability, that represents $\succcurlyeq$.[7] The latter follows from results of Debreu (1954).

2.3 Additive Separability Over Decumulative Probabilities

In this subsection we present a separability or independence property that is shared by EU, RDU and PT. It is formulated as a preference condition involving common elementary shifts in the probabilities of outcomes. Given the prospect $P \in \mathcal{L}$ we denote the prospect resulting from an elementary shift of probability ε from outcome x_i to the adjacent outcome x_{i+1} in P as the prospect

$$\varepsilon_{i,i+1}P := (p_1, \ldots, p_i - \varepsilon, p_{i+1} + \varepsilon, p_{i+2}, \ldots, p_n).$$

Whenever we use this notation, it is implicitly assumed that $p_i \geq \varepsilon > 0$ and $i \in \{1, \ldots, n-1\}$. Similarly, we write $\varepsilon_{i+1,i}P$ for the prospect that results from an elementary probability shift of ε from outcome x_{i+1} to outcome x_i in P (whereby $p_{i+1} \geq \varepsilon > 0$, $i \in \{1, \ldots, n-1\}$ is implicit in this notation). In general, we write $\varepsilon_{i,j}P$ for a (not necessarily elementary) shift of probability from outcome x_i to x_j of prospect P.

Expected utility satisfies the following property of invariance of the preferences under common elementary probability shifts.

Independence: The preference relation $\succcurlyeq$ satisfies *independence of (common elementary) probability shifts* (IPS)

$$P \succcurlyeq Q \Leftrightarrow \varepsilon_{i,i+1}P \succcurlyeq \varepsilon_{i,i+1}Q,$$

[6] The ρ-probability mixture of P with R is the prospect $\rho P + (1-\rho)R = (\rho p_1 + (1-\rho)r_1, \ldots, \rho p_n + (1-\rho)r_n)$.
[7] This function may be unbounded at x_n or x_1.

whenever $P, Q, \varepsilon_{i,i+1}P, \varepsilon_{i,i+1}Q \in \mathcal{L}$.

We demonstrate that IPS is necessary for EU. Substitution of Eq. (1) in the preceding equivalence gives

$$P \succcurlyeq Q \Leftrightarrow \sum_{j=1}^{n} p_j u(x_j) \geq \sum_{j=1}^{n} q_j u(x_j).$$

Adding $\varepsilon[u(x_{i+1}) - u(x_i)]$ to both sides of the latter inequality, one obtains the equivalence $\varepsilon_{i,i+1}P \succcurlyeq \varepsilon_{i,i+1}Q$, whenever $P, Q, \varepsilon_{i,i+1}P, \varepsilon_{i,i+1}Q \in \mathcal{L}$. Sufficiency of IPS, in the presence of weak order, first order stochastic dominance and J-continuity, has been shown in Webb and Zank (2011, Theorem 5).

RDU and PT generally violate IPS. However, they satisfy a restricted version of the principle:

Comonotonic Independence: The preference relation $\succcurlyeq$ satisfies *comonotonic independence of (common elementary) probability shifts* (CIS) if

$$P \succcurlyeq Q \Leftrightarrow \varepsilon_{i,i+1}P \succcurlyeq \varepsilon_{i,i+1}Q,$$

whenever $P, Q, \varepsilon_{i,i+1}P, \varepsilon_{i,i+1}Q \in \mathcal{L}$ such that $\sum_{j=1}^{i} p_j = \sum_{j=1}^{i} q_j$.

CIS says that common elementary probability shifts maintain the preference between two prospects if the two prospects offer identical "good news" probabilities of obtaining outcome x_i or better. That is, the decumulative probabilities of obtaining x_i or a better outcome is the same in both prospects. Obviously, this is equivalent to saying that the cumulative probability of obtaining x_{i+1} or a worse outcome is the same in both prospects, so they have identical "bad news" probabilities. Therefore, CIS requires that elementary shifts in probabilities between common decumulative probabilities of outcomes are permitted. If one writes prospects as (de)cumulative distributions over X, one immediately observes that this CIS translates into an independence requirement on a rank-ordered or comonotonic set of probability distributions, hence the name for CIS. Substitution of RDU from Eq. (2) into the preceding equivalence gives

$$\begin{aligned} P &\succcurlyeq Q \\ \Leftrightarrow \sum_{j=1}^{n}[w(p_1+\cdots+p_j)-w(p_1+\cdots+p_{j-1})]u(x_j) &\geq \sum_{j=1}^{n}[w(q_1+\cdots+q_j)-w(q_1+\cdots+q_{j-1})]u(x_j). \end{aligned}$$

This inequality remains unaffected if to both sides we add $[w(\alpha+p_{i+1}+\varepsilon))-w(\alpha+p_{i+1})]u(x_{i+1})$ and subtract $[w(\alpha)-w(\alpha-\varepsilon)]u(x_i)$, where $\alpha := \sum_{j=1}^{i} p_j = \sum_{j=1}^{i} q_j$ is set. Thus, we obtain the equivalence to $\varepsilon_{i,i+1}P \succcurlyeq \varepsilon_{i,i+1}Q$, whenever $P, Q, \varepsilon_{i,i+1}P, \varepsilon_{i,i+1}Q \in \mathcal{L}$ such that $\sum_{j=1}^{i} p_j = \sum_{j=1}^{i} q_j$.

The preceding calculations show that CIS is necessary for RDU. Similarly, it can be shown that CIS is necessary for PT. Both models require additional properties in order to distinguish them. However, CIS and the preference conditions in the previous subsection, imply that an additive separability property across outcomes holds for the representing function V. The result is formulated next and its proof follows from results of Wakker (1993) for additive representations on comonotonic sets.

Lemma 1 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{L}$:*

(i) The preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by an additive function

$$V(P) = \sum_{j=1}^{n-1} V_j(\sum_{i=1}^{j} p_i), \tag{4}$$

with continuous strictly increasing functions $V_1, \ldots, V_{n-1} : [0,1] \to \mathbb{R}$ which are bounded with the exception of V_1 and V_{n-1} which could be unbounded at extreme decumulative probabilities (i.e., V_1 may be unbounded at 1 and V_{n-1} may be unbounded at 0).

(ii) The preference relation $\succcurlyeq$ is a Jensen-continuous weak order that satisfies first order stochastic dominance and comonotonic independence of common elementary probability shifts.

The functions $V_1, \ldots, V_{n-1}$ are jointly cardinal, that is, they are unique up to multiplication by a common positive constant and addition of a real number. □

Next we focus on the condition that, if added to Lemma 1, delivers general PT. We present this principle in the next Section.

3 Consistent Probability Midpoints

In this section we present consistency requirements for elicited probability midpoints. To motivate the term "probability midpoint", suppose we have two prospects P, Q over outcomes $\{x_1, x_2, x_3, x_4\}$. Let $P = (\alpha, 0, 1-p-\alpha, p)$ and $Q = (\beta, 0, 1-q-\beta, q)$ with $\alpha < \beta$ such that $P \sim Q$. A probability shift of $\beta-\alpha$ from x_3 to x_1 in prospect P requires a shift of probability $\gamma-\beta$ from x_3 to x_1 in prospect Q in order to obtain indifference between the resulting prospects. Thus, we obtain $(\beta-\alpha)_{3,1}P \sim (\gamma-\beta)_{3,1}Q$. The conditions presented in the previous section ensure that such prospects P, Q and probability γ exist if α and β (and, therefore, p and q) are sufficiently close. Figure 1 below illustrates these indifferences in the probability triangle with outcomes x_1, x_3 and x_4.

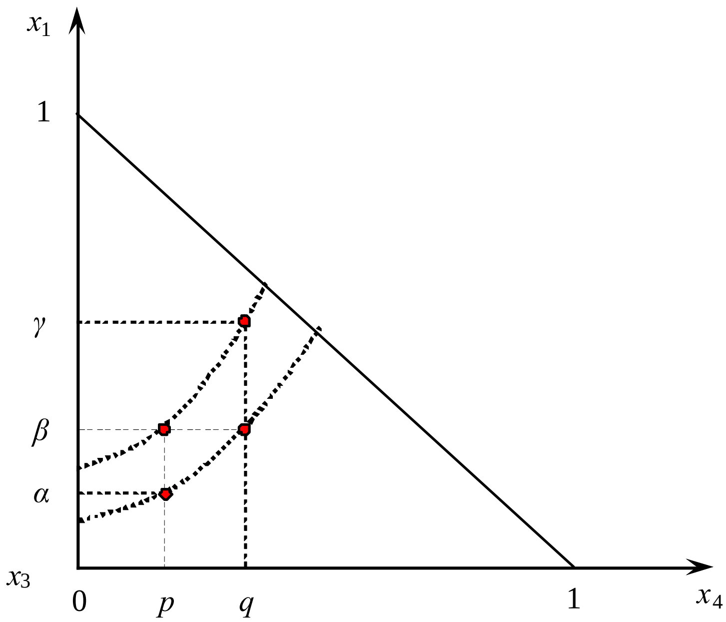

Figure 1: Elicited probability midpoint β.

Substituting the additive representation in Eq. (4) into the preceding two indifferences, we obtain

$$P \sim Q \Leftrightarrow V_1(\alpha) + V_2(\alpha) + V_3(1-p) = V_1(\beta) + V_2(\beta) + V_3(1-q)$$

and

$$(\beta - \alpha)_{3,1}P \sim (\gamma - \beta)_{3,1}Q \Leftrightarrow V_1(\beta) + V_2(\beta) + V_3(1-p) = V_1(\gamma) + V_2(\gamma) + V_3(1-q).$$

Taking the difference between the resulting equations implies

$$[V_1(\gamma) + V_2(\gamma)] - [V_1(\beta) + V_2(\beta)] = [V_1(\beta) + V_2(\beta)] - [V_1(\alpha) + V_2(\alpha)].$$

Thus, β is a probability midpoint between α and γ for the sum of functions $V_1 + V_2$ of Eq. (4).

Suppose that we know more about preferences, specifically, assume that the preference is a PT-preference and that x_3 is a gain (Case 1). Then substitution of Eq. (2) into the indifferences $P \sim Q$ and $(\beta - \alpha)_{3,1}P \sim (\gamma - \beta)_{3,1}Q$, subtraction of the second equation from the first, and cancellation of common terms give

$$w^+(\beta) - w^+(\alpha) = w^+(\gamma) - w^+(\beta). \tag{5}$$

That is, β is a probability midpoint between α and γ for the probability weighting function for gains w^+ (see Figure 2).

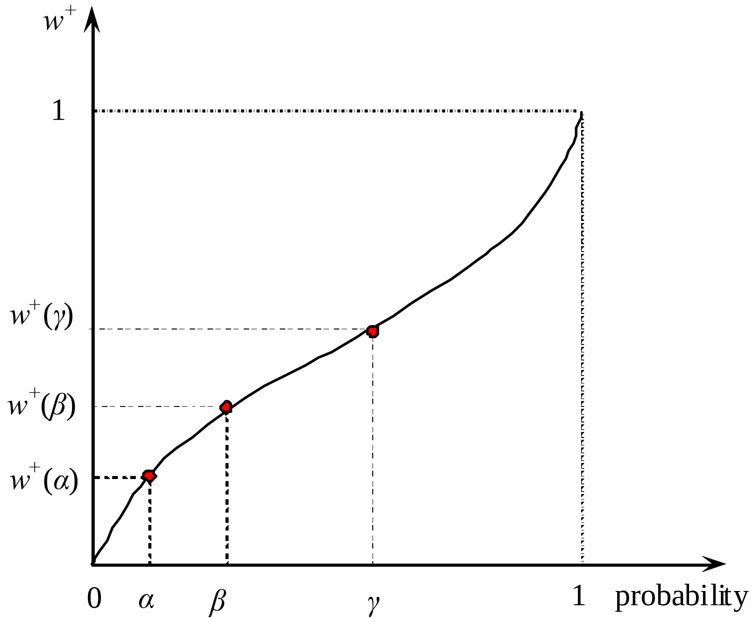

Figure 2: Equally spaced good news probabilities.

In practice one elicits midpoints by fixing the probabilities α, p, q and asking for the probability β that makes a person indifferent between prospects P and Q. Figure 3 presents such an elicitation question.

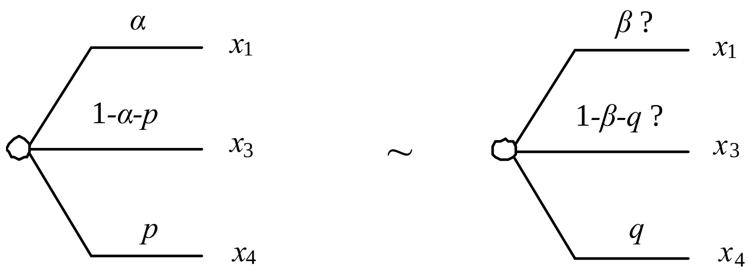

Figure 3: Eliciting standard sequences of probabilities.

Figure 3 indicates that replacing p in the left prospect with q (thus, shifting probability $q - p$ from x_3 to x_4) requires some appropriate probability being shifted from x_3 to x_1 in the prospect on the right in order to obtain indifference. The required probability shift from x_3 to x_1 is then found to be $\beta - \alpha$. Subsequently, α is replaced by β in the left prospect and one asks for the probability mass that needs to be shifted from x_3 to x_1 in order to maintain the indifference. This way one obtains $\gamma - \beta$. Continuing with this elicitation process, behavior reveals a standard sequence of equally spaced probabilities based on the the initial probability shift $q - p$ from x_3 to x_4 as unit of measurement.

Sequences of elicited probability midpoints are not meaningful unless they are independent of the outcomes used to elicit the sequence and the measurement unit $q - p$. Therefore, consistency in measuring such standard sequences is required. For example, instead of shifting $\beta - \alpha$ and $\gamma - \beta$ from x_3 to x_1, shifting the same probabilities from x_3 to x_2 should also leave the indifference unaffected. That is, $(\beta - \alpha)_{3,2}P \sim (\gamma - \beta)_{3,2}Q$ should be obtained. Indeed, Eq. (5) follows from substitution of PT in $P \sim Q$ and in $(\beta - \alpha)_{3,2}P \sim (\gamma - \beta)_{3,2}Q$, subtraction of the second equation from the first, and cancellation of common terms.

Next we continue our analysis but assume that x_2 is a gain while x_3 is the reference point (Case 2) or a loss (Case 3). In Case 2 the indifferences $P \sim Q$ and $(\beta - \alpha)_{3,1}P \sim (\gamma - \beta)_{3,1}Q$, substitution of PT into these indifferences, subtraction of the second equation from the first and cancellation of common terms, give

$$[w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)]u(x_1) = 0.$$

The latter holds only if β is a probability midpoint between α and γ for w^+. The same conclusion is obtained if PT is substituted into $P \sim Q$ and $(\beta - \alpha)_{3,2}P \sim (\gamma - \beta)_{3,2}Q$. Thus, consistency in probability shifts is obtained.

Let us now turn to Case 3 (x_2 is a gain and x_3 is a loss). Then, substituting PT into the indifferences $P \sim Q$ and $(\beta - \alpha)_{3,1}P \sim (\gamma - \beta)_{3,1}Q$ implies

$$[w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)]u(x_1) = [w^-(1 - \alpha) - 2w^-(1 - \beta) + w^-(1 - \gamma)]u(x_3)$$

and substituting PT in the second pair of indifferences $P \sim Q$ and $(\beta-\alpha)_{3,2}P \sim (\gamma-\beta)_{3,2}Q$ implies

$$[w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)]u(x_2) = [w^-(1-\alpha) - 2w^-(1-\beta) + w^-(1-\gamma)]u(x_3). \tag{6}$$

Combining the two equations we obtain

$$[w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)]u(x_1) = [w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)]u(x_2),$$

which holds only if $[w^+(\alpha) - 2w^+(\beta) + w^+(\gamma)] = 0$ (as $u(x_1) > u(x_2)$ is assumed). Equivalently, this means that β is a probability midpoint between α and γ for w^+. But then, substitution into Eq. (6) says that $1-\beta$ is a probability midpoint between $1-\gamma$ and $1-\alpha$ for the probability weighting function w^-. Reformulated in terms of the dual of w^- it means that β is a probability midpoint between α and γ for $\hat{w}^-$. If this holds for all elicited midpoints, then sign-dependence becomes meaningless and preferences are represented by RDU.

For Cases 1 and 2 we have concluded that genuine PT (that is, PT with sign-dependence) and the requirement of consistency in probability shifts leads to a meaningful statement of β being a probability midpoint between α and γ for w^+ independent of outcomes, as long as x_3 is a gain or the reference point. When x_3 is a loss, sign-dependence and consistency cannot hold jointly. Our first property states the consistency requirement for general prospects but without a priori knowledge of whether we have sign-dependence.

Good News Midpoint Consistency: The preference relation $\succcurlyeq$ satisfies *consistency in probability midpoints* above x_i or *good news midpoint consistency* (GMC) at $x_i, i \in \{2, \ldots, n\}$ if

$$\begin{aligned} P = (\alpha, 0, \ldots, 0, p_i, \ldots, p_n) \quad &\sim \quad Q = (\beta, 0, \ldots, 0, q_i, \ldots, q_n) \\ \text{and } (\beta-\alpha)_{i,1}P \quad &\sim \quad (\gamma-\beta)_{i,1}Q \\ \text{imply } (\beta-\alpha)_{i,j}P \quad &\sim \quad (\gamma-\beta)_{i,j}Q, \end{aligned}$$

for all $j \in \{1, \dots, i-1\}$ whenever $\alpha < \beta < \gamma$ are probabilities such that $P, Q, (\beta - \alpha)_{i,1}P$, and $(\gamma - \beta)_{i,1}Q$ are well-defined.[8]

It can be verified that RDU satisfies GMC at x_i for all $i = 2, \dots, n$. This has been shown in Zank (2010). Further, RDU also satisfies a property, dual to GMC, defined next.

Bad News Midpoint Consistency: The preference relation $\succcurlyeq$ satisfies *consistency in probability midpoints* at x_i or *bad news midpoint consistency* (BMC) at $x_i, i \in \{1, \dots, n-1\}$ if

$$\begin{aligned} P = (p_1, \dots, p_i, 0, \dots, 0, \alpha) &\sim Q = (q_1, \dots, q_i, 0, \dots, 0, \beta) \\ \text{and } (\beta - \alpha)_{i,n}P &\sim (\gamma - \beta)_{i,n}Q \\ \text{imply } (\beta - \alpha)_{i,j}P &\sim (\gamma - \beta)_{i,j}Q, \end{aligned}$$

for all $j \in \{i+1, \dots, n\}$ whenever $\alpha < \beta < \gamma$ are probabilities such that $P, Q, (\beta - \alpha)_{i,n}P$, and $(\gamma - \beta)_{i,n}Q$ are well-defined.[9]

In contrast to RDU, genuine PT does satisfy GMC at x_i only for $i \in \{2, \dots, k\}$ and it satisfies BMC at x_i only for $i \in \{k, \dots, n-1\}$. Unless PT-preferences agree with RDU, there are no further outcomes, except the reference point x_k, where both GMC and BMC hold. Usually, in applications of PT we do not know the reference point. However, the preceding consistency properties for probability midpoints can serve as a test for detecting at which outcome one of the properties fails. Thus, GMC and BMC provide a critical test for PT-preferences through sign-dependence for elicited probability midpoints. We build this test into the next preference condition.

Sign-Dependent Midpoint Consistency: The preference relation $\succcurlyeq$ satisfies *sign-dependent probability midpoint consistency* (SMC) if for each outcome x_i, $i \in \{2, \dots, n-1\}$ the preference satisfies good news midpoint consistency at x_i or bad news midpoint consistency at x_i (or

[8] In this definition we have included the case $i = 2$, which holds trivially, for completeness.

[9] Similar to GMC, in this definition we have included the case $i = n-1$, which holds trivially, for completeness.

both).

Let us look at the implications of SMC. First, consider that $M \in \{3, \ldots, n-1\}$ is such that GMC holds at x_M. Let M be maximal with this property. That is, there is no $j > M$ such that GMC holds at x_j. If $M \leq n-1$ then BMC holds at x_i for all $i = M, \ldots, n-1$. Let m be minimal with the property that BMC holds at x_m. Two cases can occur:

(i) $M = m$, in which case we have sign-dependence. Then we can set $k := M$, and x_k is a (unique) reference point;

(ii) $m < M$, in which case we have no sign-dependence, thus no reference point.

Suppose now that $M = 2$ and there is no $j > M$ such that GMC holds at x_M. Then BMC holds at x_i for all $i = M, \ldots, n-1$. Then we obtain Case (i) above with $M = m = 2$ and sign-dependence. Therefore, either we have sign-dependence and a unique reference point, or we do not have sign-dependence, and hence no reference point. As our main result below shows, Case (ii) gives RDU, the special case of PT without reference dependence, and if $3 \leq M = m \leq n-2$ then Case (i) gives genuine PT.

The cases $M = m = 2$ and $M = m = n-1$, however, warrant special attention. The reason for this is the possible unboundedness of the functions V_1 at 1 and of V_{n-1} at 0 as stated in Lemma 1. For example, representing functionals of the following form are compatible with all preference conditions presented above:

$$W(P) = V_1(p_1) + \sum_{j=2}^{n} [w^-(p_j + \cdots + p_n) - w^-(p_{j+1} + \cdots + p_n)]u(x_j), \tag{7}$$

where u and w^- are as in PT and V_1 converging to ∞ at 1 is as in Lemma 1 above, or

$$W(P) = \sum_{j=1}^{n-2} [w^+(p_1 + \cdots + p_j) - w^+(p_1 + \cdots + p_{j-1})]u(x_j) + V_{n-1}(1 - p_{n-1}), \tag{8}$$

where u and w^+ are as in PT and V_{n-1} converging to $-\infty$ at 0 (i.e., when p_{n-1} approaches 1) is as in Lemma 1 above.

Specifically one can choose $V_1(p_1) = [\frac{p_1}{1-p_1}]u(x_1)$ and $V_{n-1}(1-p_n) = [\frac{1-p_n}{p_n}]u(x_n)$ in Eqs. (7) and (8) above. In the first case (Eq. (7)) one can think of a patient who has been diagnosed with a severe disease, such as cancer. The various available treatments can lead to outcomes of which the best one is $x_1 =$ "healed". It should be obvious that this outcome is so attractive that any treatment with even the smallest positive probability for x_1 will be superior to any other treatment that has zero probability for x_1. Another related example is documented in Thaler and Johnson (1990) and analyzed in Barberis, Huang and Santos (2001). After having faced a series of losses, many investors attempt to break even by taking additional risks despite the chances to break even being small. By contrast, the second representation above (Eq. (8)), can be thought of an extreme form of pessimism, where the possible loss x_n is extremely unattractive and any prospect with positive likelihood for x_n will be regarded as inferior to a prospect with zero probability for x_n. Individuals exhibiting this form of pessimism are willing to buy insurance at prices far above the actuarially fair value to avoid the loss x_n.

Behavior described above with extreme optimism for a good outcome or extreme pessimism for a bad outcome is not excluded here. Instead, we allow for such preferences and we refer to the resulting representations as "generalized" prospect theory, that is, PT including the cases of $k = 2$ and V_1 in Lemma 1 unbounded, or $k = n - 1$ and V_{n-1} in Lemma 1 unbounded. We can now present our main result.

THEOREM 2 *The following two statements are equivalent for a preference relation $\succcurlyeq$ on $\mathcal{L}$:*

(i) The preference relation $\succcurlyeq$ on $\mathcal{L}$ is represented by generalized prospect theory, with the functions V_1 or V_{n-1} in Lemma 1 possibly unbounded.

(ii) The preference relation $\succcurlyeq$ is a Jensen-continuous weak order that satisfies first order stochas-

tic dominance, comonotonic independence of common elementary probability shifts, and sign-dependent probability midpoint consistency.

Whenever V_1 and V_{n-1} are bounded, the probability weighting functions are uniquely determined. If further $w^+ \neq \hat{w}^-$, the reference point is unique and the utility function is a ratio scale; otherwise, if $w^+ = \hat{w}^-$, utility is cardinal. If V_1 (or V_{n-1}) is unbounded then w^- (w^+) is uniquely determined and V_1 (or V_{n-1}) and u are jointly cardinal with $u(x_k) = 0$ restricting the location parameter of u to 0. $\square$

4 Extensions

In the previous sections we have assumed that we have strictly ordered outcomes. The strict ordering can be relaxed if there are at least four strictly ordered outcomes. If X is finite all results remain valid if we take representatives for each set of indifferent outcomes. These outcomes will then be given the same utility value. If, however, X is infinite, then results remain valid for each finite subset of outcomes Y that contains at least four outcomes that are strictly ordered. We can then extend the PT-representations on the sets of prospects over the different finite subsets, Y and Y', to a general PT-representation by using the fact that the representations on any such sets of prospects over Y and of prospects over Y' must agree with the representation on the set of prospects over $Y \cup Y'$. Hence a common PT-representation must exist over prospects with finite support in the possibly infinite X.

If there are only three strictly ordered outcomes, the sign-dependent probability midpoint consistency principle is trivially satisfied. In that case, we require stronger tools to obtain additive separability (Lemma 1). We can still derive an additive representation by using stronger conditions like the Thomsen condition or triple cancellation as in Wakker (1993, Theorem 3.2). Those additive functions can be seen as the product of utility times the corresponding weighting function and we immediately obtain generalized PT. To obtain the special case of RDU, we have to additionally invoke the probability tradeoff consistency principle of Abdellaoui (2002) or a refinement of that principle as proposed in Köbberling and Wakker (2003). For fewer than three strictly ordered outcomes first order stochastic dominance and weak order are sufficient for an ordinal representation of preferences.

In our derivation of PT it has been essential that the weighting functions are continuous at 0 and at 1. Discontinuities at these extreme probabilities are, however, empirically meaningful. We could adopt a weaker version of Jensen-continuity that is restricted to prospects that have common best and worst outcomes with positive objective probability. Such conditions have been used in Cohen (1992) and more recently in Webb and Zank (2011) where probability weighting functions are derived that are linear and discontinuous at extreme probabilities. These weighting functions can then be described by two parameters one for optimism and one for pessimism. As Webb and Zank show, this relaxation of continuity in probabilities comes at a price. They require additional structural assumptions for the preference in order to obtain consistency of the parameters across sets of prospects with different minimal and maximal outcomes. Also, specific consistency principles that imply the uniqueness of these parameters are required. We conjecture that in our framework such consistency principles can be formulated for nonlinear weighting functions that are discontinuous at 0 and at 1. A formal derivation of PT with such weighting functions is, however, beyond the scope of this paper.

5 Conclusion

The focus of this paper has been on sign-dependence, the different treatment of probabilities depending on whether the latter are attached to gains or to losses. We have complemented existing foundations for PT in the "continuous utility approach" with preference foundations based on the "continuous weighting function approach" by adopting and extending a familiar tool from empirical measurement of probability weighting functions, the midpoint consistency principle. Preference midpoints for outcomes are a useful tool for the analysis of risk attitudes captured by utility. It was recently shown by Baillon, Driessen and Wakker (2012), how these midpoint based tools facilitate the analysis of ambiguity preferences and time preferences. We have demonstrated how similar midpoint tools can be adapted for the analysis of PT-preferences. Our method facilitates the analysis of probabilistic risk attitudes and, therefore, complements the utility-based approach. Further, we have shown how the probability midpoint principle can be employed to identify reference points in an efficient and tractable manner.

Appendix: Proofs

As can be observed from the equations for RDU, PT and Eqs. (7) and (8), and also from the additively separable preference representation in Lemma 1 very frequently we use cumulated probabilities as variables. For the proofs it will be convenient to use an alternative notation for prospects, following Abdellaoui (2002) and Zank (2010). In the *decumulative probabilities* notation $P = (\tilde{p}_1, \ldots, \tilde{p}_n)$, where $\tilde{p}_j = \sum_{i=1}^{j} p_i$ denotes the probability of obtaining outcome x_j or better, $j = 1, \ldots, n$.[10] Obviously, $\tilde{p}_n = 1$. Naturally, all preference conditions can be re-written in terms of decumulative probabilities.

Proof of Lemma 1: The proof of the lemma follows from results for additive representations on rank-ordered sets in Wakker (1993, Theorem 3.2 and Corollary 3.6). That statement (i) implies statement (ii) is immediate from the properties of the functions $V_j, j = \{1, \ldots, n-1\}$. As we have a preference relation $\succcurlyeq$ defined on a rank-ordered set of decumulative probabilities (i.e., a rank-ordered subset of $[0,1]^{n-1}$) and $\succcurlyeq$ satisfies weak order, Jensen-continuity and first order stochastic dominance, we also have Euclidean continuity (by Lemma 18 in Abdellaoui 2002) for $\succcurlyeq$. First order stochastic dominance comes down to strong monotonicity in decumulative probabilities. Further, as $n \geq 4$, and our independence of common elementary probability shifts comes down to coordinate independence of Wakker (1993), statement (ii) of Theorem 3.2 of Wakker is satisfied. Then statement (i) of the lemma follows from statement (i) of Theorem 3.2 of Wakker, the only difference being that our strong monotonicity implies that the functions $V_j, j = \{1, \ldots, n-1\}$ are strictly increasing. Uniqueness results are as in Wakker's Theorem 3.2. This concludes the proof of Lemma 1. $\square$

Proof of Theorem 2: The derivation of statement (ii) from statement (i) follows from Lemma 1 and the analysis preceding the theorem in the main text on the consistency of elicited probability

[10] Similarly, in the *cumulative probabilities* notation $P = (1, 1 - \tilde{p}_1, \ldots, 1 - \tilde{p}_{n-1})$ where entries denote the probability of obtaining outcome x_j or less, $j = 1, \ldots, n$.

midpoints under PT.

We now prove that statement (ii) implies statement (i) of the theorem. Assume that $\succcurlyeq$ on $\mathcal{L}$ is a weak order that satisfies first order stochastic dominance, independence of common elementary probability shifts and sign-dependent probability midpoint consistency. Then, by statement (i) of Lemma 1 the preference $\succcurlyeq$ on $\mathcal{L}$ is represented by an additive function

$$V(P) = \sum_{j=1}^{n-1} V_j(\tilde{p}_j), \tag{9}$$

with continuous strictly increasing functions $V_1, \ldots, V_{n-1} : [0,1] \to \mathbb{R}$ which are bounded except V_1 and V_{n-1} which could be unbounded at extreme probabilities.

Next we restrict our analysis to decumulative probabilities different from 0 or 1 to, for now, avoid the problems with the unboundedness of V_1 and V_{n-1}. Following the analysis in the main text preceding Theorem 2, SMC implies that either there is no sign-dependence or we have a unique reference point $x_k, k \in \{2, \ldots, n-1\}$. If we do not have sign-dependence, SMC comes down to the consistency in probability attitudes of Zank (2010), which implies that RDU holds. Therefore, we consider the case that we have sign-dependence.

Assume first that $2 < k \leq n-1$. For any $\delta \in (0,1)$ and $\varepsilon > 0$ let $B_\varepsilon(\delta)$ be the open neighborhood around δ with Euclidean distance ε. Take any $\alpha, \beta, \gamma \in B_\varepsilon(\delta)$ such that

$$\sum_{i=1}^{k-1}[V_i(\beta) - V_i(\alpha)] = \sum_{i=1}^{k-1}[V_i(\gamma) - V_i(\beta)]. \tag{10}$$

For sufficiently small $\varepsilon > 0$, by continuity of the functions $V_i, i = k, \ldots, n-1$, there exists lotteries $P, Q \in \mathcal{L}$ with

$$\sum_{i=1}^{k-1} V_i(\alpha) + \sum_{i=k}^{n-1} V_i(\tilde{p}_i) = \sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(\tilde{q}_i)$$

and

$$\sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(\tilde{p}_i) = \sum_{i=1}^{k-1} V_i(\gamma) + \sum_{i=k}^{n-1} V_i(\tilde{q}_i).$$

Before we proceed with the proof we introduce some simplifying notation. For any nonempty subset $I \subset \{1, \ldots, n-1\}$ we write $\sigma_I P$ for prospect P with $\tilde{p}_i$ replaced by $\sigma \in [0,1]$ for all $i \in I$. Clearly, for $\sigma_I P$ to be a well-defined prospect, I must include all indices between and including the smallest ($\min\{i : i \in I\}$) and the largest ($\max\{i : i \in I\}$) in I. With this notation, the latter two equations are equivalent to the respective indifferences

$$\alpha_I P \sim \beta_I Q \text{ and } \beta_I P \sim \gamma_I Q,$$

where $I = \{1, \ldots, k-1\}$, meaning that the decumulative probabilities α, β, γ are attached to gains. Consider the case $\alpha < \beta$ (and note that the case $\alpha > \beta$ is completely analogous). By first order stochastic dominance it follows that $\gamma > \beta$. Further, sign-dependent probability midpoint consistency requires that

$$\alpha_J \beta_{I \backslash J} P \sim \beta_J \gamma_{I \backslash J} Q$$

for all $J = \{1, \ldots, j\}, j \in I \backslash \{k-1\}$. First take $j = 1$. Then, substitution of Equation (9) into $\alpha_I P \sim \beta_I Q$ implies

$$\sum_{i=1}^{k-1} V_i(\alpha) + \sum_{i=k}^{n-1} V_i(\tilde{p}_i) = \sum_{i=1}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(\tilde{q}_i),$$

and substitution of Equation (9) into $\alpha_1 \beta_{I \backslash \{1\}} P \sim \beta_1 \gamma_{I \backslash \{1\}} Q$ gives

$$V_1(\alpha) + \sum_{i=2}^{k-1} V_i(\beta) + \sum_{i=k}^{n-1} V_i(\tilde{p}_i) = V_1(\beta) + \sum_{i=2}^{k-1} V_i(\gamma) + \sum_{i=k}^{n-1} V_i(\tilde{q}_i).$$

Taking the difference of the two latter equations and cancelling common terms implies

$$\sum_{i=2}^{k-1}[V_i(\beta) - V_i(\alpha)] = \sum_{i=2}^{k-1}[V_i(\gamma) - V_i(\beta)].$$

Similarly, joint substitution of Equation (9) into $\beta_I P \sim \gamma_I Q$ and $\alpha_1\beta_{I\setminus\{1\}}P \sim \beta_1\gamma_{I\setminus\{1\}}Q$, taking differences and cancelling common terms, imply

$$V_1(\beta) - V_1(\alpha) = V_1(\gamma) - V_1(\beta). \tag{11}$$

Similarly, if $j = 2$, we obtain

$$\sum_{i=3}^{k-1}[V_i(\beta) - V_i(\alpha)] = \sum_{i=3}^{k-1}[V_i(\gamma) - V_i(\beta)]$$

and

$$\sum_{i=1}^{2}[V_i(\beta) - V_i(\alpha)] = \sum_{i=1}^{2}[V_i(\gamma) - V_i(\beta)],$$

and using Equation (11) we obtain

$$V_2(\beta) - V_2(\alpha) = V_2(\gamma) - V_2(\beta).$$

By induction on j we conclude that if Equation (10) holds then for all $j = 1, \ldots, k-1$ we have

$$V_j(\beta) - V_j(\alpha) = V_j(\gamma) - V_j(\beta).$$

That the converse holds is immediate. We conclude that for any $\delta \in (0,1)$ and sufficiently small $\varepsilon > 0$

for $\alpha, \beta, \gamma \in B_\varepsilon(\delta)$ we have

$$\begin{aligned}\sum_{i=1}^{k-1}[V_i(\beta) - V_i(\alpha)] &= \sum_{i=1}^{k-1}[V_i(\gamma) - V_i(\beta)] \\ &\Leftrightarrow \\ V_j(\beta) - V_j(\alpha) &= V_j(\gamma) - V_j(\beta) \text{ for all } j = 1, \ldots, k-1.\end{aligned}$$

This means that locally the functions V_j, $j = 1, \ldots, k-1$, are proportional and also proportional to their sum, which we denote V^+. Global proportionality follows from local proportionality and continuity. It follows that positive constants $s_1, \ldots, s_{k-1}$ and real numbers $t_1, \ldots, t_{k-1}$ exist such that

$$V_j(\cdot) = s_j V^+(\cdot) + t_j, j = 1, \ldots, k-1.$$

Following Proposition 3.5 of Wakker (1993) the functions V_j can be taken finite at 0 and 1, and can continuously be extended to all of $[0, 1]$.

Similar arguments, now applying consistency for probability midpoints of losses, can be used to derive proportionality of the functions $V^- := \sum_{j=k}^{n-1} V_j$ and V_j, $j = k, \ldots, n-1$ whenever $2 \leq k < n-1$. Proposition 3.5 of Wakker (1993) applies again saying that the functions V_j can be taken finite at 0 and 1, and can continuously be extended to all of $[0, 1]$. We conclude that positive constants $s_k, \ldots, s_{n-1}$ and real numbers $t_k, \ldots, t_{n-1}$ exist such that

$$V_j(\cdot) = s_j V^-(\cdot) + t_j, j = k, \ldots, n-1.$$

Next, for the case that $2 < k < n-1$, we derive the weighting functions for probabilities of gains and losses and the utility for outcomes. We fix $V^+(1) + V^-(1) = 1$ and $V_j(0) = 0$ for $j = 1, \ldots, k-1$ and $V_j(1) = 0$ for $j = k, \ldots, n-1$, thereby fixing the scale and location of the otherwise jointly cardinal functions V_j. Then, $t_1 = \cdots = t_{n-1} = 0$ must hold and it follows that $V^+(1) = 1$. We define

$$w^+(\tilde{p}) := V^+(\tilde{p}) = \sum_{j=1}^{k-1} V_j(\tilde{p}) + \sum_{j=k}^{n-1} V_j(1).$$

Therefore, $w^+(0) = 0, w^+(1) = 1$ and w^+ is strictly increasing and continuous on $[0, 1]$, and is, indeed, a well-defined probability weighting function. It is the probability weighting function for probabilities of gains.

Next we derive w^-. First we define

$$\hat{w}(\tilde{p}) := \frac{V^-(\tilde{p})}{V^-(0)} = \frac{\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(\tilde{p})}{\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(0)}.$$

This is a well-defined function given that the V_j's, $j = k, \ldots, n-1$, are strictly increasing and bounded, and thus, $V_j(\tilde{p}) < 0$ for all $j = k, \ldots, n-1$, whenever $\tilde{p} < 1$, such that the denominator $\sum_{j=1}^{k-1} V_j(0) + \sum_{j=k}^{n-1} V_j(0) \neq 0$ and finite. It follows that the function $\hat{w}$ has the following properties: $\hat{w}(1) = 0$ and $\hat{w}(0) = 1$ and $\hat{w}$ is strictly decreasing and continuous on $[0, 1]$. We set

$$\tilde{w}^-(\tilde{p}) := 1 - \hat{w}(\tilde{p}) = \frac{V^-(0) - V^-(\tilde{p})}{V^-(0)},$$

for each $\tilde{p} \in [0, 1]$, which gives us the dual weighting function for probabilities of losses. A useful rearrangement of this equation gives

$$V^-(\tilde{p}) = V^-(0)[1 - \tilde{w}^-(\tilde{p})].$$

From $\tilde{w}^-$ we obtain w^- through $w^-(\tilde{p}) = 1 - \tilde{w}^-(1 - \tilde{p})$ for all $\tilde{p} \in [0, 1]$.

Next we derive the utility function for outcomes. From the derivation of w^+ and $V_j(\cdot) = s_j V^+(\cdot), j = 1, \ldots, k-1$, we obtain

$$V_j(\cdot) = s_j w^+(\cdot), j = 1, \ldots, k-1,$$

and from the derivation of $\tilde{w}^-$ and $V_j(\cdot) = s_j V^-(\cdot), j = k, \ldots, n-1$, we obtain

$$V_j(\cdot) = s_j V^-(0)[1 - \tilde{w}^-(\cdot)], j = k, \ldots, n-1.$$

Noting that the degenerate prospect that gives x_i for sure is expressed as the prospect $0_{\{1,\ldots,i-1\}}(1, \ldots, 1)$, for each $i = 1, \ldots, n$, we define utility as follows:

$$\begin{aligned} u(x_k) := & \quad V(0_{\{1,\ldots,k-1\}}(1, \ldots, 1)) \\ & = V^-(1) \\ & = 0. \end{aligned}$$

Moving backwards, for $i = k-1, \ldots, 1$ we iteratively define

$$u(x_i) := u(x_{i+1}) + s_i.$$

And for $i = k+1, \ldots, n$ we iteratively define

$$u(x_i) := u(x_{i-1}) + s_{i-1} V^-(0).$$

These definitions of utility for gains and losses imply that the ordering of the utility for outcomes is $u(x_1) > \cdots > u(x_n)$, thus in agreement with the ordering according to the preference $\succcurlyeq$.

Next, substitution into $V(P)$, gives

$$\begin{aligned} V(P) &= \sum_{j=1}^{k-1} V_j(\tilde{p}_j) + \sum_{j=k}^{n-1} V_j(\tilde{p}_j) \\ &= \sum_{j=1}^{k-1} s_j w^+(\tilde{p}_j) + \sum_{j=k}^{n-1} s_j V^-(0)[1 - \tilde{w}^-(\tilde{p}_j)]. \end{aligned}$$

Note that for $i = 1, \ldots, k-1$ we have $s_i = u(x_i) - u(x_{i+1})$ and for $i = k, \ldots, n-1$ we have $s_i V^-(0) = u(x_{i+1}) - u(x_i)$. Substitution into the preceding equation gives

$$\begin{aligned} V(P) &= \sum_{j=1}^{k-1} w^+(\tilde{p}_j)[u(x_j) - u(x_{j+1})] + \sum_{j=k}^{n-1} [u(x_{j+1}) - u(x_j)][1 - \tilde{w}^-(\tilde{p}_j)] \\ &= \sum_{j=1}^{k-1} [w^+(\tilde{p}_j) - w^+(\tilde{p}_{j-1})]u(x_j) + \sum_{j=k}^{n-1} [u(x_{j+1}) - u(x_j)][1 - \tilde{w}^-(\tilde{p}_j)], \end{aligned}$$

where, in the latter equation, the term relating to probabilities of gains has been rearranged using the properties that $w^+(\tilde{p}_{j-1}) = 0$ for $j = 1$ and $u(x_{j+1}) = 0$ for $j = k-1$. Next we rearrange the term relating to probabilities of losses. After substitution of $\tilde{w}^-(p) = 1 - w^-(1-p)$, we obtain

$$V(P) = \sum_{j=1}^{k-1} [w^+(\tilde{p}_j) - w^+(\tilde{p}_{j-1})]u(x_j) + \sum_{j=k+1}^{n} [u(x_j) - u(x_{j-1})]w^-(1 - \tilde{p}_{j-1}).$$

Rearranging we obtain

$$V(P) = \sum_{j=1}^{k-1} [w^+(\tilde{p}_j) - w^+(\tilde{p}_{j-1})]u(x_j) + \sum_{j=k+1}^{n} [w^-(1 - \tilde{p}_{j-1}) - w^-(1 - \tilde{p}_j)]u(x_j) = PT(P), \tag{12}$$

where in the derivation of the latter expression for loss probabilities we have used that $u(x_{j-1}) = 0$ for $j = k+1$ and $w^-(1 - \tilde{p}_j) = 0$ for $j = n$ (recall that $\tilde{p}_n = 1$). We conclude that the representation V of $\succcurlyeq$ on $\mathcal{L}$ is, in fact, a genuine PT-functional.

Cases $k = 2$ and $k = n-1$ are problematic as we may have unboundedness at 0 or 1. If $k = 2$ and V_1 is bounded at 1, or if $k = n-1$ and V_{n-1} is bounded at 0, we can simply repeat the preceding analysis and derive genuine PT. If, however, $k = 2$ and V_1 is unbounded at 1, we can derive w^- and u for losses (i.e., for $x_i, i = 3, \ldots, n$) by using similar arguments as in the preceding analysis (i.e., following the case $2 \le k < n-1$); nothing more can be said about V_1, thus, generalized PT as in Eq. (8) is obtained. Similarly, if $k = n-1$ and V_{n-1} is unbounded at 0, we can derive w^+ and u for gains (i.e., for $x_i, i = 1, \ldots, n-1$) by using similar arguments as in the preceding analysis (i.e., following the case $2 < k \leq n-1$), thus, generalized PT as in Eq. (7) is obtained. Hence, generalized PT represents $\succcurlyeq$ on $\mathcal{L}$. This concludes the derivation of statement (i) from statement (ii) in Theorem 2.

To complete the proof of the theorem we need to derive the uniqueness results for the weighting functions and utility. For bounded V_1 and V_{n-1} we have fixed the scale and location of the otherwise jointly cardinal functions $V_j, j = 1, \ldots, n-1$ in order to derive w^+ and w^-. That is, given any other representation of preferences that is additively separable as in Lemma 1, fixing scale and location as required in the proof above will lead to the same probability weighting functions. This shows that the weighting functions w^+ and w^- are uniquely determined. From the definition of the utility function u it is clear that the only freedom we have in defining utility is the starting value at the reference point x_k (i.e., the location parameter) and a scaling parameter due to the jointly cardinal functions $V_j, j = 1, \ldots, n-1$. So, utility can, at most, be cardinal. However, in order to rewrite V in the form of the PT-functional it is critical that $u(x_k) = 0$. Otherwise, if $u(x_k) \neq 0$, the terms $w^+(p_k)u(x_k)$ and $u(x_k)w^-(1-p_k)$ will appear in Equation (12). With these terms added in Equation (12) a functional is obtained that violates first order stochastic dominance and continuity, hence, it cannot be a representation of $\succcurlyeq$ on $\mathcal{L}$. This means that u must be a ratio scale. This is somewhat different if one of V_1 or V_{n-1} are unbounded. In the first case w^- is uniquely determined but u, which must satisfy $u(x_2) = 0$, and V_1 are jointly cardinal. In the second case w^+ is uniquely determined and u, which must satisfy $u(x_{n-1}) = 0$, and V_{n-1} are jointly cardinal.

This concludes the proof of Theorem 2. □

References

Abdellaoui, Mohammed, 2000. Parameter-Free Elicitation of Utility and Probability Weighting Functions. Manage. Sci. 46, 1497–1512.

Abdellaoui, Mohammed, 2002. A Genuine Rank-Dependent Generalization of the von Neumann-Morgenstern Expected Utility Theorem. Econometrica 70, 717–736.

Abdellaoui, Mohammed, Olivier l'Haridon and Horst Zank, 2010. Separating Curvature and Elevation: A Parametric Probability Weighting Function. J. Risk Uncertainty 41, 39–65.

Abdellaoui, Mohammed, Frank Vossmann, and Martin Weber, 2005. Choice-based Elicitation and Decomposition of Decision Weights for Gains and Losses under Uncertainty. Manage. Sci. 51, 1384–1399.

Andreoni, James and Charles Sprenger, 2009. Certain and Uncertain Utility: The Allais Paradox and Five Decision Theory Phenomena. Working Paper UCSD.

Andreoni, James and Charles Sprenger, 2012. Risk Preferences Are Not Time Preferences. Amer. Econ. Rev. 102, 3357–3376.

Baillon, Aurélien, Bram Driesen, and Peter P. Wakker, 2012. Relative Concave Utility for Risk and Ambiguity. Games Econ. Behav. 75, 481–489.

Barberis Nicholas, Ming Huang and Tano Santos, 2001. Prospect Theory and Asset Prices. Quart. J. Econ. 116 1–53.

Bateman, Ian J., Alistair Munro, Bruce Rhodes, Chris Starmer and Robert Sugden, 1997. A Test of the Theory of Reference-Dependent Preferences. Quart. J. Econ. 112, 647–661.

Bell, David E., 1985. Disappointment in Decision Making under Uncertainty. Oper. Res. 33, 1–27.

Birnbaum, Michael H. and Steven E. Stegner, 1981. Measuring the Importance of Cues in Judgment for Individuals: Subjective Theories of IQ as a Function of Heredity and Environment. J. Exper. Social Psych. 17, 159–182.

Bleichrodt, Han and Jose Luis Pinto, 2000. A Parameter-Free Elicitation of the Probability Weighting Function in Medical Decision Analysis. Manage. Sci. 46, 1485–1496.

Bleichrodt, Han, Jose Luis Pinto and Peter P. Wakker, 2001. Making Descriptive Use of Prospect Theory to Improve the Prescriptive Use of Expected Utility. Manage. Sci. 47, 1498–1514.

Bleichrodt, Han, Ulrich Schmidt, and Horst Zank, 2009. Additive Utility in Prospect Theory. Manage. Sci. 55, 863–873.

Chateauneuf, Alain, 1999. Comonotonicity Axioms and Rank-Dependent Expected Utility for Arbitrary Consequences. J. Math. Econ. 32, 21–45.

Chateauneuf, Alain, Jürgen Eichberger and Simon Grant, 2007. Choice under Uncertainty with the Best and Worst in Mind: NEO-Additive Capacities. J. Econ. Theory 137, 538–567.

Chateauneuf, Alain and Peter P. Wakker, 1999. An Axiomatization of Cumulative Prospect Theory for Decision under Risk. J. Risk Uncertainty 18, 137–145.

Cohen, Michèle, 1992. Security Level, Potential Level, Expected Utility: A Three-Criteria Decision Model under Risk. Theory Dec. 33, 101–134.

Debreu, Gérard, 1954. Representation of a Preference Ordering by a Numerical Function. In Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis (Eds.) Decision Processes, 159–165, Wiley, New York.

Diecidue, Enrico, Ulrich Schmidt, and Horst Zank, 2009. Parametric Weighting Functions. J. Econ. Theory 144, 1102–1118.

Edwards, Ward, 1953. Probability-Preferences in Gambling. Amer. J. Psych. 66, 349–364.

Edwards, Ward, 1954. Probability-Preferences Among Bets With Differing Expected Values. Amer. J. Psych. 67, 56–67.

Etchart-Vincent, Nathalie, 2004. Is Probability Weighting Sensitive to the Magnitude of Consequences? An Experimental Investigation on Losses. J. Risk Uncertainty 28, 217–235.

Hogarth, Robin M. and Hillel J. Einhorn, 1990. Venture Theory: A Model of Decision Weights. Manage. Sci. 36, 780–803.

Kahneman, Daniel, Jack L. Knetsch and Richard H. Thaler, 1990. Experimental Tests of the Endowment Effect and the Coase Theorem. J Polit. Econ. 98, 1325–1348.

Kahneman, Daniel and Amos Tversky, 1979. Prospect Theory: An Analysis of Decision under Risk. Econometrica 47, 263–291.

Kahneman, Daniel and Amos Tversky, 2000. Choices, Values, and Frames. Cambridge University Press, New York.

Köbberling, Veronika and Peter P. Wakker, 2003. Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique. Math. Operations Res. 28, 395–423.

Köbberling, Veronika and Peter P. Wakker, 2004. A Simple Tool for Qualitatively Testing, Quantitatively Measuring, and Normatively Justifying Savage's Subjective Expected Utility. J. Risk Uncertainty 28, 135–145.

Kőszegi, Botond, and Matthew Rabin, 2006. A model of reference-dependent preferences. Quart J. Econ 121, 1133–1165.

Kőszegi, Botond, 2010. Utility from anticipation and personal equilibrium. Econ Theory 44, 415–444.

Kothiyal, Amit, Vitalie Spinu, and Peter P. Wakker, 2011. Prospect Theory for Continuous Distributions: A Preference Foundation. J. Risk Uncertainty 42, 195–210.

Loewenstein, George F. and Daniel Adler, 1995. A Bias in the Prediction of Tastes. Econ. J. 105, 929–937.

Payne, John W., 2005. It is Whether You Win or Lose: The Importance of the Overall Probabilities of Winning or Losing in Risky Choice. J. Risk Uncertainty 30, 5–19.

Prelec, Drazen, 1998. The Probability Weighting Function. Econometrica 66, 497-527.

Quiggin, John, 1981. Risk Perception and Risk Aversion among Australian Farmers. Australian J. Agric. Econ. 25, 160–169.

Quiggin, John, 1982. A Theory of Anticipated Utility. J Econ. Behav. Org. 3, 323–343.

Samuelson, William F. and Richard J. Zeckhauser, 1988. Status Quo Bias in Decision Making. J. Risk Uncertainty 1, 7–59.

Schmeidler, David, 1989. Subjective Probability and Expected Utility without Additivity. Econometrica 57, 571–587.

Schmidt, Ulrich and Horst Zank, 2009. A Simple Model of Cumulative Prospect Theory. J. Math. Econ. 45, 308–319.

Schmidt, Ulrich and Horst Zank, 2012. A Genuine Foundation for Prospect Theory. J. Risk Uncertainty 45, 97–113.

Shalev, Johnathan, 2000. Loss aversion equilibrium. Inter J. Game Theory 29, 269–287

Shalev, Johnathan, 2002. Loss Aversion and Bargaining. Theory Dec 52, 201–232.

Starmer, Chris, 2000. Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk. J. Econ. Literature 38, 332–382.

Thaler, Richard H., 1980. Toward a Positive Theory of Consumer Choice. J. Econ. Behav. Org. 1, 39–60.

Thaler, Richard H. and Eric J. Johnson, 1990. Gambling with the House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice. Manage. Sci. 36, 643–660.

Tversky, Amos and Daniel Kahneman, 1992. Advances in Prospect Theory: Cumulative Representation of Uncertainty. J. Risk Uncertainty 5, 297–323.

van de Kuilen, Gijs, and Peter P. Wakker, 2011. The Midweight Method to Measure Attitudes toward Risk and Ambiguity. Manage. Sci. 57, 582–598.

Viscusi, Kip W., Wesley A. Magat, and Joel Huber, 1987. An Investigation of the Rationality of Consumer Valuations of Multiple Health Risks. Rand J. Econ. 18, 465–479.

Viscusi, W. Kip and Joel C. Huber, 2012. Reference-Dependent Valuations of Risk: Why Willingness-to-Accept Exceeds Willingness-to-Pay. J. Risk Uncertainty 44, 19–44.

von Neumann, John and Oskar Morgenstern (1944, 1947, 1953) Theory of Games and Economic Behavior. Princeton University Press, Princeton NJ.

Wakker, Peter P., 1989. Additive Representations of Preferences, A New Foundation of Decision Analysis. Kluwer Academic Publishers, Dordrecht.

Wakker, Peter P., 1993. Additive Representations on Rank-Ordered Sets II. The Topological Approach. J. Math. Econ. 22, 1–26.

Wakker, Peter P., 1994. Separating Marginal Utility and Probabilistic Risk Aversion. Theory Dec. 36, 1–44.

Wakker, Peter P., 2001. Testing and Characterizing Properties of Nonadditive Measures through Violations of the Sure-Thing Principle. Econometrica 69, 1039–1059.

Wakker, Peter P., 2010. Prospect Theory for Risk and Ambiguity. Cambridge University Press, Cambridge, UK.

Wakker, Peter P. and Amos Tversky, 1993. An Axiomatization of Cumulative Prospect Theory. J. Risk Uncertainty 7, 147–176.

Wakker, Peter P. and Horst Zank, 2002. A Simple Preference-foundation of Cumulative Prospect Theory with Power Utility. Europ. Econ. Rev. 46, 1253–1271.

Webb, Craig S. and Horst Zank, 2011. Accounting for Optimism and Pessimism in Expected Utility. J. Math. Econ. 47, 706–717.

Wu, George, Jiao Zhang, Mohammed Abdellaoui, 2005. Testing Prospect Theories using Probability Tradeoff Consistency. J. Risk Uncertainty 30, 107–131.

Zank, Horst, 2001. Cumulative Prospect Theory for Parametric and Multiattribute Utilities. Math. Operations Res. 26, 67–81.

Zank, Horst, 2010. Consistent Probability Attitudes. Econ. Theory 44, 167–185.