The Distribution of Unit Root Test Statistics after Seasonal Adjustment

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Abstract

This paper argues that seasonal adjustment will generally induce noninvertible moving averages and examines the consequences for the distributions of (zero frequency) unit root test statistics for I(1) and near-integrated processes. The seasonal adjustment procedure analyzed is the two-sided X-11 seasonal adjustment filter, which generally leads to a high order MA component in the adjusted series. As standard unit root tests do not satisfactorily deal with high order noninvertible MAs, inferences (even asymptotically) about the presence of unit roots can be unreliable for seasonally adjusted data. We illustrate these effects analytically and through Monte Carlo simulation, for the Augmented Dickey-Fuller, Phillips-Perron, the M-type and the Variance Ratio unit root tests applied with a variety of detrending procedures.

Keywords: Unit root tests, seasonality, seasonal adjustment, X-11, noninvertible moving averages, Near Integrated proceses.

JEL codes: C22, C12, C82.

1 Introduction

Seasonal adjustment is routinely applied to macroeconomic time series by official statistical agencies around the world, with these adjusted values then employed in the analyses undertaken by commentators and researchers. It is not surprising, therefore, that many authors have studied whether seasonal adjustment distorts the underlying properties of time series; Wallis (1974) and Sims (1974) are pioneering works on this topic, while more recent analyses include del Barrio Castro and Osborn (2004), Ericsson, Hendry and Tran (1994), Ghysels and Perron (1993), Ghysels and Liebermann (1996), and Matas-Mir, Osborn and Lombardi (2008). Although these studies establish nontrivial consequences for seasonal adjustment in terms of shortrun properties, the general conclusion with respect to longrun properties is reassuring, with seasonal adjustment found to have no asymptotic impact on tests under the null hypothesis of (zero frequency) integration and cointegration; see, in particular, Ghysels and Perron (1993) and Ericsson *et al.* (1994).

These reassuring results about longrun properties, however, rest on an invertibility assumption that may be invalid for seasonally adjusted data. To be more specific, due to the nature of the filters embedded in the commonly applied methods of X-11 and X-12-ARIMA, seasonal adjustment using these procedures will generally give rise to noninvertible moving average terms in the adjusted data; see Maravall (1993) and Bell (2012) for details. Therefore, if unit root tests do not satisfactorily deal with noninvertible moving average components, then inferences (even asymptotically) about the presence of unit roots can be unreliable for seasonally adjusted data.

The present paper studies the consequences of seasonal adjustment for a range of unit root tests, using both analytical and Monte Carlo simulation methods. In addition to the parametric Augmented Dickey-Fuller test (Dickey and Fuller, 1979) [ADF], we examine the Phillips-Perron (1988) test [PP], the Variance Ratio test of Breitung (2002) [VRT] and the M-type tests ([MZ_α], [MZ_t], [MSB]) proposed by Stock (1999) and popularized by Perron and Ng (1996). The analysis we undertake is related to that of Galbraith and Zinde-Walsh (1999), who examine the impact of moving average components on ADF tests. However, in contrast to their assumption of invertibility, we focus on the noninvertible moving average case and, more specifically, on the effect of seasonal adjustment. Also, although Ghysels and Perron (1993) examine the impact of seasonal adjustment on unit root tests, they assume invertibility.

Indeed, the issue we study has not, to our knowledge, been considered previously in the literature. Maravall (1993) discusses the noninvertibility implication of seasonal adjustment, and hence recommends that unit root tests based on autoregressive augmentation should not be undertaken with seasonally adjusted data. However, he does not analyze the resulting asymptotic distributions. Although the Monte Carlo analyses of Ghysels (1990), Ghysels and Perron (1993) and Smith and Otero (2002) indicate size problems for univariate unit root or cointegration tests after seasonal adjustment, this is seen to be a finite sample issue. In contrast, we examine analytically the affect on the asymptotic distributions of unit root tests, then considering finite sample results in the light of those findings.

The paper is organized as follows. Section 2 contains some general discussion of seasonal adjustment and unit roots. Section 3 then analytically examines the ADF, PP, VRT, MZ_α, MZ_t and MSB (zero frequency) unit root tests in the presence of seasonal adjustment under near integration, followed by a Monte Carlo study in Section 4. Section 5 concludes.

2 Seasonal Adjustment and Unit Roots

This section provides background discussion of seasonal adjustment, focussing on its implication of noninvertibility at seasonal frequencies, and unit root tests.

2.1 The X-11 symmetric filter

Virtually all seasonal adjustment procedures are based on an unobserved components specification, for which the additive version is

$$y_t^u = S_t + T_t + I_t \tag{1}$$

where y_t^u is the unadjusted (observed) time series, while S_t, T_t and I_t are the seasonal, trend and irregular components respectively. The components S_t and T_t may be deterministic or, more typically, stochastic, with the three components assumed to be driven by mutually independent processes.

As described by Wallis (1974), Ghysels and Osborn (2001, chapter 4) and others, seasonal adjustment by X-11 involves the application of a sequence of linear filters[1]. Therefore, the seasonally adjusted, or filtered, series y_t^f can be represented as

$$y_t^f = y_t^u - \widehat{S}_t = \widehat{T}_t + \widehat{I}_t = q(L)y_t^u \tag{2}$$

where L is the usual lag operator. Except for observations near the beginning and end of the sample period, the filter $q(L)$ is a two-sided symmetric filter of the form

$$q(L) = \sum_{i=k}^{-k} q_i L^i \tag{3}$$

in which $q_i = q_{-i}$, $i = 1, ..., k$. Wallis (1974) graphically shows the implied linear coefficients for the X-11 filter, with Laroque (1977) and Ghysels and Perron (1993) tabulating these for the quarterly and monthly data, respectively. Although the filter weights q_i sum to unity, they are not all positive. It is also noteworthy that while the linear filter is an approximation to X-11, this captures the essential features of the seasonal adjustment procedure (Ghysels and Osborn, 2001, pp. 100-101). Also, while X-11 has largely been replaced by X-12-ARIMA, and more recently X-13ARIMA-SEATS (U.S. Census Bureau, 2013), the symmetric X-11 filter retains its core role for seasonal adjustment; see Findley *et al.* (1998), the discussion in Ghysels and Osborn (2001, pp. 106-108) and the documentation in U.S. Census Bureau (2013).

The extent of the two-sided symmetric filter $q(L)$, represented by k in (3), is about seven years in either direction (Wallis, 1974). Observations towards the beginning and end of the available sample (when sufficient values are not available for (3) to be employed) are seasonally adjusted in X-12-ARIMA using asymmetric filters and/or forecasting and backcasting (see Findley *et al.*, 1998). Nevertheless, the two-sided filter $q(L)$ provides the essential element of seasonal adjustment through these procedures and we follow previous researchers in focussing on the implications of this filter.

Bell (2012) documents the unit root properties of seasonal adjustment filters; in particular, his Lemma 1 catalogues those for the two-sided X-11 symmetric linear filter and implies the estimated seasonal component $\widehat{S}_t$ has the form

$$\begin{aligned} \widehat{S}_t &= (1-L)^3(1-L^{-1})^3\omega(L)y_t^u \\ &= -L^{-4}(1-L)^6\omega(L)y_t^u \end{aligned} \tag{4}$$

where $\omega(L)$ is a linear two-sided filter. In practice observed macroeconomic series contain at most two zero frequency autoregressive (AR) unit roots, but the transformation used to obtain $\widehat{S}_t$ in (4) employs sixth order differencing. In terms of the seasonally adjusted series itself, Bell (2012) further establishes (using our notation) that the two-sided filter can be written as

$$q(L) = U(L)(1+L^{-1})\omega^*(L) = L^{-1}U(L)(1+L)\omega^*(L) \tag{5}$$

where $U(L)$ is the moving annual summation operator $U(L) = 1 + L + \cdots + L^{s-1}$, s is the frequency of observations per year ($s = 4$ or 12 for quarterly or monthly data, respectively) and $\omega^*(L)$ is another two-sided polynomial in L. Indeed, Bell (2012, p.458) further notes that, to a good approximation,

$$q(L) \approx U(L)U(L^{-1})\omega^\dagger(L) = L^{-s+1}[U(L)]^2\omega^\dagger(L) \tag{6}$$

[1] This is true for the additive form of (1) with default options and consequently also for the form which is additive after taking logarithms. The multiplicative version of X-11 is also viewed as approximately linear; see Bell (2012).

where $\omega^{\dagger}(L)$ is again a two-sided filter.

The factor $U(L)$ in (5) implies that the filter annihilates the full set of AR unit roots at seasonal frequencies if these are present in y_t^u, in addition to annihilating deterministic seasonal effects. However, even if y_t^u is seasonally integrated with an AR polynomial containing $U(L)$, the additional smoothing factor $(1 + L)$ leads the adjusted series to be noninvertible unless y_t^u contains two seasonal AR unit roots at the Nyquist frequency. Indeed, (6) further implies that the X-11 filter effectively annihilates not just a single set of unit roots at all seasonal frequencies, but two such sets. Since observed series typically do not contain even one full set AR seasonal unit roots (see, among others Beaulieu and Miron, 1993, Osborn, 1990, or the discussion in Ghysels and Osborn, 2001, pp.90-91), the MA component of adjusted series in practice will (at least approximately) include one or more unit roots at all seasonal frequencies.

The general message is that time series seasonally adjusted using X-11 will be noninvertible at seasonal frequencies, possibly with multiple unit roots. Although our focus is on the X-11 filter, it is noteworth that the use of model-based adjustment (for example, using the TRAMO-SEATS procedure of Gomez and Maravall, 1997, or within X-13ARIMA-SEATS) does not avoid this issue. Indeed, Bell (2012) also shows that symmetric finite model-based adjustment filters contain the exact factor $U(L)U(L^{-1}) = [U(L)]^2$, so that issues arising from noninvertibility after seasonal adjustment apply across all commonly applied seasonal adjustment procedures.

2.2 Unit root tests

The regression

$$y_t = \varphi y_{t-1} + v_t \tag{7}$$

is the basis of all zero frequency unit root tests, with the relevant null hypothesis $\varphi = 1$ or, equivalently, $\alpha = \varphi - 1 = 0$. The disturbance innovations v_t in (7) may exhibit temporal dependence and/or heteroskedasticity, with the limiting distribution of the normalized bias and t-ratio statistics for testing this null hypothesis given by Phillips (1987, Theorem 3.1). As shown by Phillips (1987), the distribution of tests for $\varphi = 1$ in (7) depends on unknown parameters related to the serial correlation of the innovations. The two widely used approaches proposed to deal with this problem are those of Dickey and Fuller (1979), which deals with serial correlation by augmenting the test regression (7) with lagged differences of y_t, and the nonparametric serial correlation correction methodology of Phillips (1987) and Phillips and Perron (1988).

The seminal study of Schwert (1989) found *ADF* unit root tests to be poorly sized when v_t in (7) has an MA component that nearly cancels with the zero frequency unit root. The analyses of Galbraith and Zinde-Walsh (1999) and Gonzalo and Pitarakis (1998) show why such distortions occur. In particular, these studies establish the dependence of the size distortions on the order of augmentation adopted, so that (with finite augmentation) distortions exist even asymptotically. However, while Galbraith and Zinde-Walsh (1999) and Gonzalo and Pitarakis (1998) analytically examine the implications of MA components in the disturbance of (7), both assume these to be invertible. Nevertheless, Galbraith and Zinde-Walsh (1999) hint at the importance of this assumption, by noting that the size distortions in the ADF test are particularly difficult to deal with in the presence of a near-noninvertible MA root.

In contrast to the ADF test, which relies on an AR approximation to a MA, the PP approach uses observed residuals from (7) to mimic the autocorrelation properties of v_t, typically up to some maximum lag. Provided that the value employed for this maximum lag is sufficiently large, this approach is particularly attractive in the context of MA processes.

The typical assumption in unit root analyses (for example, Ghysels and Perron, 1993, Elliot, Rothenberg and Stock, 1996, Galbraith and Zinde-Walsh, 1999) is that the process for v_t is stationary and invertible. However, it is obvious that application of the linear X-11 filter to an unadjusted series in (7) yields

$$y_t^f = \varphi y_{t-1}^f + q(L)v_t \tag{8}$$

and hence, from the discussion of the preceding subsection, the transformed disturbances will contain one or more MA unit roots at seasonal frequencies. This paper focuses on the unstudied issue of the impact of a noninvertible moving average component on the distributions of conventional zero frequency unit root tests, both under the null hypothesis and for near-integrated cases. Our analysis includes not only the ADF test, which may have particular difficulties in this context since it requires approximating the noninvertible MA by an AR augmentation, but also PP tests and the M-statistic variant of the PP approach proposed by Stock (1999), both of which effectively adopt an MA representation. Nevertheless, these latter tests require estimation of the long-run variance of $q(L)v_t$ in (8), which in turn implies that its noninvertible nature be captured adequately. Finally, we examine the variance ratio approach of Breitung (2002) which does not require estimation of any nuisance parameters.

3 Asymptotic Distributions

This section discusses the near-integrated processes that we analyze, and then presents results for the asymptotic distributions for a range of unit root tests applied to the seasonally adjusted series.

3.1 Zero frequency near-integrated processes

To illustrate the implications of seasonal adjustment, we assume that the true data generating process (DGP) for the original, unadjusted, data series (y_t^u) is the near-integrated nonseasonal process

$$\begin{aligned} y_t^u &= \exp\left(\frac{c}{T}\right) y_{t-1}^u + \epsilon_t, \qquad t = 1, 2, ..., T \\ \varphi_T &= \exp\left(\frac{c}{T}\right) \approx \left(1 + \frac{c}{T}\right) \end{aligned} \tag{9}$$

where, for simplicity, we assume that the ϵ_t innovations follow $\epsilon_t \sim iid\left(0, \sigma_\epsilon^2\right)$. However, it is possible to extend the results presented here allowing weak dependence in ϵ_t without altering the qualitative conclusions drawn from the analysis of (9). When $c = 0$, (9) is a random walk process. Although it may appear unrealistic to seasonally adjust such a nonseasonal process, nevertheless it is informative because all the autocorrelation characteristics of this $I(1)$ process are induced by adjustment, therefore allowing us to focus on the impact of the filter. Further, analysis of this process maximizes the extent of noninvertibility induced by the filter, and in this sense represents a "worst case" scenario. When c is "small" and $c < 0$, the process of (9) is near-integrated at the zero frequency, so that our analysis covers this a DGP of this form, in addition to $I(1)$ processes.

At this point it is important to consider the nature of the starting value y_1^u. As in Elliott, Rothenberg and Stock (1996), one possible assumption is that it satisfies

$T^{-1/2} y_1^u \to 0$, for which a specific example is the so-called conditional case. However, the arguments of Pantula, Gonzalez-Farias and Fuller (1994) are persuasive that the conditional case is not tenable for macroeconomic data. Hence we follow their proposal and employ the unconditional case, whereby y_1^u follows the same DGP as the other sample observations. More precisely, we assume that $y_1^u = \sum_{i=0}^{[\lambda T]+k} \varphi_T^i \epsilon_{1-i}$ for $\lambda \geq 0$ and in which $k > 0$ is the length of the seasonal adjustment filter in (3); see (for example) Canjels and Watson (1997) and Rodrigues and Taylor (2004) for similar starting value assumptions. Note that we introduce k in the upper limit of the summation in order to facilitate the starting value assumption made below for the filtered observations; Rodrigues and Taylor (2004) employ a corresponding assumption when working with seasonal data.

We analyze the effect of linear X-11 seasonal adjustment filter, as widely used in practice and discussed in the previous section. Consequently, the process after adjustment is given by

$$y_t^f = \exp\left(\frac{c}{T}\right) y_{t-1}^f + u_t, \qquad t = 1, 2, ..., T \tag{10}$$

where, from (3), $u_t = \sum_{i=-k}^{k} q_{|i|} \epsilon_{t+i} = q(L)\epsilon_t$. Although in practice observations y_t^f for $t = 1, ..., T$ are computed from unfiltered values from periods $1-k$ to $T+k$, we treat (9) as a DGP. For this process, we make the following assumptions:

Assumption A.1 The error term u_t $(t = 1, ..., T)$ is generated as $u_t = \sum_{i=-k}^{k} q_{|i|} \epsilon_{t+i} = q(L)\epsilon_t$, with $\epsilon_t \sim iid\left(0, \sigma_\epsilon^2\right)$ for $t = -k+2, ..., T+k$.

Assumption A.2 The initial condition is given by $y_1^f = \sum_{i=0}^{[\lambda T]} \varphi_T^i u_{1-i}$ with $\lambda > 0$.

Note that Assumption A.1 requires the *iid* process for ϵ_t to extend for k periods prior to and after the sample period for y_t^f, in order to take account of the two-sided filter at each end of the sample. Assumption A.2 is the unconditional starting value assumption as made by Canjels and Watson (1997), and is compatible with the $k-1$ pre-sample unadjusted observations being of analogous form to that assumed for y_1^u.

As discussed in subsection 2.1, the X-11 seasonal adjustment filter applied to (9) results in the presence of seasonal unit roots in the MA of u_t in (10). Clearly, therefore, the filtered process retains the near-AR unit root of (9), but is distorted through the complicated and noninvertible moving average introduced in the disturbances u_t. Using the Beveridge-Nelson (1981) decomposition, the filtered series can be written (see the Appendix) as:

$$\begin{aligned} y_t^f &= \varphi_T^{t-1} y_1^f + \sum_{j=0}^{t-2} \varphi_T^j u_{t-j} \\ &= \varphi_T^{t-1} y_1^f + \sum_{j=0}^{t-2} \varphi_T^j q(\varphi_T^{-1}) \epsilon_{t-j} + \sum_{j=1}^{k} \epsilon_{t+j} \sum_{i=j}^{k} q_i \varphi_T^{i-j} + \sum_{j=1}^{k} \epsilon_{2-j} \sum_{i=j}^{k} q_i \varphi_T^{t-2+k-i} \\ &\quad - \sum_{j=0}^{k-1} \epsilon_{t-j} \sum_{i=j+1}^{k} q_i \varphi_T^{j-i} - \sum_{j=0}^{k-1} \epsilon_{2+j} \sum_{i=j+1}^{k} q_i \varphi_T^{t-2+i} \end{aligned} \tag{11}$$

for $t = 2, ..., T$. Expression (11) allows us to obtain the distributions of the unit root test statistics.

We analyze the effect of seasonal adjustment in the local to unity framework for the ADF and PP tests, together with the nonparametric approaches of the M-type (MZ_α, MZ_t, MSB) and VRT tests. The case the ADF test with no augmentation is considered in the next subsection, which illustrates the implications of the autocorrelation induced by the filter, with the usual AR augmentation allowed in subsection 3.3, while the PP and other tests are considered in subsequent subsections.

3.2 No correction for autocorrelation

Application of the DF test regression (7) to the filtered process of (10) yields

$$\begin{aligned} T\left(\hat{\varphi}_T - \varphi_T\right) &= \frac{T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} u_t}{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^2} \\ t_{(\hat{\varphi}_T - \varphi_T)} &= T\left(\hat{\varphi}_T - \varphi_T\right) \times \sqrt{\frac{T^{-2}\sum\limits_{t=1}^{T}\left(y_{t-1}^{f}\right)^2}{\hat{\sigma}_u^2}} \end{aligned} \tag{12}$$

where $\hat{\sigma}_u^2$ is the OLS estimator of the variance of u_t. The following proposition shows how the asymptotic distributions of the unit root test statistics depend on the X-11 filter coefficients q_i of (3); the proof of this, and subsequent propositions, can be found in the Appendix.

Proposition 1 *For an unadjusted series following the near integrated process of (9) to which the symmetric linear filter of (3) is applied, then*

$$T\left(\hat{\varphi}_T - 1\right) \Rightarrow \frac{\int_0^1 J_c(\lambda, r)\, dW(r) + \frac{1}{2}\left[1 - \sum\limits_{i=-k}^{k} q_i^2\right]}{\int_0^1 J_c(\lambda, r)^2\, dr} \tag{13}$$

while the asymptotic distribution for the t-ratio statistic is

$$t_{(\hat{\varphi}_T - 1)} \Rightarrow c\sqrt{\int_0^1 J_c(\lambda, r)^2\, dr}\sqrt{\sum_{i=-k}^{k} q_i^2} + \frac{\int_0^1 J_c(\lambda, r)\, dW(r) + \frac{1}{2}\left[1 - \sum\limits_{i=-k}^{k} q_i^2\right]}{\sqrt{\int_0^1 J_c(\lambda, r)^2\, dr}\sqrt{\sum\limits_{i=-k}^{k} q_i^2}}. \tag{14}$$

Here, and throughout the paper, $\Rightarrow$ indicates convergence in distribution, $J_c(\lambda, r)$ is an Ornstein-Uhlenbeck process and $W(r)$ is a standard Brownian motion defined in the appendix.

The distribution of the normalized bias in (13) is shifted to the right compared with the usual DF near-integrated case, due to the presence of the numerator term $\left[1 - \sum\limits_{i=-k}^{k} q_i^2\right]$, which is equal to 0.174 and 0.214 for quarterly and monthly data, respectively. In the case of the distribution of the t-ratio, (14) is affected not only by the numerator term as in (13), but also by a scaling effect of the square root of the sum of the squared filter weights. This latter effect is also substantial, being 0.909 and 0.887 in the quarterly and monthly cases, respectively. Overall, under the null hypothesis ($c = 0$)

we anticipate that the seasonally adjusted random walk process will result in undersized DF test statistics when no allowance is made for the autocorrelation in this process. An important issue, therefore, is the extent to which allowing for autocorrelation mitigates these effects.

3.3 Autoregressive augmentation

Now consider the more realistic case where the ADF regression

$$\Delta y_t^f = \alpha y_{t-1}^f + \sum_{i=1}^{p} \phi_i^p \Delta y_{t-i}^f + e_t^p \tag{15}$$

is applied to the filtered series of (10), when the DGP is again the near-integrated process (9). Note also that in (15) $\alpha = c/T$ as $\left(1 - \left[1 + \frac{c}{T}\right] L\right) = \Delta + (c/T)L$ and all coefficients are indexed by p, the order of augmentation employed. The MA process induced by seasonal adjustment is not fully accounted by AR augmentation. However, since the autocorrelation function of Δy_t^f with $c = 0$ can be computed from the filter coefficients of $q(L)$, then the pseudo-true coefficients ϕ_i^p $(i = 1, ..., p)$ resulting from the application of least squares to (15) are also known. More precisely, defining Γ_p as the $p \times p$ covariance matrix with i, j^{th} element given by $E[\Delta y_{t-i} \Delta y_{t-j}]$ and γ_p as the $p \times 1$ vector with i^{th} element $E[\Delta y_{t-i} \Delta y_t]$, then

$$\phi^p = \Gamma_p^{-1} \gamma_p \tag{16}$$

where $\phi^p = (\phi_1^p, \ ..., \ \phi_p^p)'$.

As shown in the Appendix, since

$$e_t^p = u_t - \sum_{i=1}^{p} \phi_i^p u_{t-i}$$

then, using $u_t = q(L)\epsilon_t$, it follows that e_t^p is a two-sided MA of the form

$$\begin{aligned} e_t^p &= \theta^p(L)\epsilon_t \\ &= (\theta_{-k}^p L^{-k} + \cdots + \theta_{-1}^p L^{-1} + \theta_0^p + \theta_1^p L + \cdots + \theta_{k+p}^p L^{k+p})\epsilon_t \end{aligned} \tag{17}$$

where k is the maximum lag of the seasonal adjustment filter in (3). For a specific data frequency (typically quarterly or monthly) and given p, the implied (asymptotic) moving average coefficients of (17) can be obtained analytically. The following proposition establishes how these MA coefficients affect the distributions of the ADF test under the null and local alternatives.

Proposition 2 *For an unadjusted series following the near integrated process of (9), the ADF regression (15) applied to the filtered series of (10) has normalized bias and t-ratio test statistics that satisfy:*

$$T\hat{\alpha} \Rightarrow c + \frac{\theta^p\left(1\right) \int_0^1 J_c\left(\lambda, r\right) dW\left(r\right) + A}{\int_0^1 J_c\left(\lambda, r\right)^2 dr} \tag{18}$$

and

$$t_{\hat{\alpha}} \Rightarrow c\sqrt{\int_0^1 J_c\left(\lambda, r\right)^2 dr}\sqrt{B} + \frac{\theta^p\left(1\right) \int_0^1 J_c\left(\lambda, r\right) dW\left(r\right) + A}{\sqrt{\int_0^1 J_c\left(\lambda, r\right)^2 dr}\sqrt{B}} \tag{19}$$

with

$$A = \theta^p(1)\sum_{i=1}^{k} q_i - \sum_{i=1}^{k}\theta^p_{-i}\sum_{j=1}^{i} q_j + \sum_{i=1}^{k+1}\theta^p_i\sum_{j=0}^{i-1} q_j + \sum_{i=k+2}^{k+p}\theta^p_i\sum_{j=0}^{k} q_j \tag{20}$$

$$B = \sum_{i=-k}^{k+p}\left(\theta^p_i\right)^2 \tag{21}$$

where θ^p_i *are defined in (17) and* $\theta^p(1) = \sum_{i=-k}^{k+p}\theta^p_i$.

Note that these results are similar to those reported by Galbraith and Zinde-Walsh (1999, Proposition 1) for the limiting null distribution of the ADF test in the presence of an invertible moving average process. In particular Galbraith and Zinde-Walsh (1999) show in their Propositions 2 and 3, that if p is such that $p = O(\delta \ln T)$, $\delta > 0$, then their scale factor tend to one and the numerator of their shift factor tend to zero. But in our situation the results of Propositions 2 and 3 in Galbraith and Zinde-Walsh (1999) do not apply, because their proof that A tends to zero and $\theta^p(1)/\sqrt{B}$ to one depends on the assumption that all roots of the polynomial $\theta^p(L) = q(L)\phi^p(L)$ lie outside the unit circle. As shown in Bell (2012) and discussed above, the filter $q(L)$ associated with X-11 seasonal adjustment has unit roots at all seasonal frequencies. Finally, also note that the results Said and Dickey (1984), Beck (1974) and Chang and Park (2002) do not apply here, because the polynomial $q(L)$ does not have a convergent infinite order autoregressive representation.

Table 1 collects the values of the shift and scale terms in (19) for different values of p (the order of augmentation) when the X-11 filter is applied to a quarterly series. The usual asymptotic distribution for near-integrated case will apply when the scalings $\sqrt{B} = 1$, $\theta^p(1)/\sqrt{B} = 1$ and shift $A/\sqrt{B} = 0$, while the null distribution for the ADF test (where $c = 0$) requires only the second and third of these to hold.

The results show that, although the X-11 filter is noninvertible, the AR augmentation takes sufficient account of the induced autocorrelation that the Dickey-Fuller asymptotic distribution applies to the t-ratio test statistic for an $I(1)$ process when a high level of augmentation is applied. This result does not appear to have been previously available, due to the invertibility assumption made in previous studies, such as Ghysels and Perron (1993). Moreover, the implication drawn by Maravall (1993) that ADF tests should not be applied to seasonally adjusted series due to noninvertibility is overly cautious.

For practical purposes, however, it is also notable that the scaling effect $\theta^p(1)/\sqrt{B}$ substantially shifts the distribution to the right when a moderately large augmentation, such as $p = 20$, is applied, with this shift further enhanced by the positive scale effect for values of p that are multiples of the seasonal frequency of 4. This points to the test being under-sized. Further, as p increases $\sqrt{B}$ declines, although not monotonically. For given c, this term indicates the power loss from using filtered data, for which the induced autocorrelation is accounted for by AR augmentation. The effects of these shift and scaling terms are further investigated in the next section.

3.4 Phillips-Perron approach

Phillips (1987) and Phillips-Perron (1988) propose correcting the normalized bias and t-ratio statistics to take account of serial correlation through the use of

$$Z(\hat{\alpha}) = T\hat{\alpha} - \frac{1}{2}\frac{\left(s_l^2 - s_u^2\right)}{T^{-2}\sum_{t=1}^{T}\left(y^f_{t-1}\right)^2} \tag{22}$$

and

$$Z\left(t_{\hat{\alpha}}\right)=\left(\frac{s_l}{s_u}\right)t_{\hat{\alpha}}-\frac{1}{2}\frac{\left(s_l^2-s_u^2\right)}{s_l\sqrt{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^f\right)^2}} \tag{23}$$

respectively, where

$$s_u^2=T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 \tag{24}$$

$$s_l^2=T^{-1}\sum_{t=1}^{T}\hat{u}_t^2+2T^{-1}\sum_{i=1}^{p}w\left(i,m\right)\sum_{t=i+1}^{T}\hat{u}_t\hat{u}_{t-i} \tag{25}$$

in which m is the truncation parameter or bandwidth, $\hat{u}_t$ $(t=1,\ldots,T)$ are the residuals from an ordinary least squares estimation of (12) with no augmentation and $w(i,m)$ is a weighting (or kernel) function used to ensure that the estimated longrun variance s_l^2 is nonnegative. Proposition 3 gives the resulting asymptotic distributions of the filtered data under near-integration.

Proposition 3 *For an unadjusted series following the near integrated process of (9), the PP test statistics of (22) and (23), applied to the seasonally adjusted series of (10), have asymptotic distributions*

$$Z\left(\hat{\alpha}\right)\Rightarrow\frac{\int_0^1 J_c\left(\lambda,r\right)dW\left(r\right)+\frac{1}{2}\left\{\left[1-\sum_{i=-k}^{k}q_i^2\right]-\left(2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k}^{k-i}q_jq_{j-i}\right)\right\}}{\int_0^1 J_c\left(\lambda,r\right)^2dr} \tag{26}$$

and

$$\begin{aligned}Z\left(t_{\hat{\alpha}}\right)\Rightarrow&\frac{\int_0^1 J_c\left(\lambda,r\right)dW\left(r\right)}{\sqrt{\int J_c\left(\lambda,r\right)^2dr}\sqrt{\sum_{i=-k}^{k}q_i^2+2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k}^{k-i}q_jq_{j-i}}}\\&+\frac{1}{2}\frac{\left[1-\sum_{i=-k}^{k}q_i^2\right]-\left(2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k+i}^{k}q_jq_{j-i}\right)}{\sqrt{\int J_c\left(\lambda,r\right)^2dr}\sqrt{\sum_{i=-k}^{k}q_i^2+2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k+i}^{k}q_jq_{j-i}}}.\end{aligned} \tag{27}$$

Note that both (26) and (27) are functions of the X-11 filter coefficients; in particular, these lead to shift and scale terms appearing in these expressions. Hence, in order for the distributions to be free asymptotically from these nuisance parameters, the shift term must tend to zero and that the scale term tend to one.

Now, for the shift term

$$\left[1-\sum_{i=-k}^{k}q_i^2\right]-\left(2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k+i}^{k}q_jq_{j-i}\right) \tag{28}$$

$1-\sum_{i=-k}^{k}q_i^2=q\left(1\right)^2-\sum_{i=-k}^{k}q_i^2=2\sum_{j=-k+1}^{k}\sum_{i=1}^{j-1}q_jq_{j-i}$ as $q\left(1\right)^2=q\left(1\right)=1$. Consequently, the choice of an appropriate bandwidth m and kernel will cause (28) to tend to zero.

Further, note that the denominator scale factor

$$\varpi = \sqrt{\sum_{i=-k}^{k} q_i^2 + 2\sum_{i=1}^{p} w\left(i,m\right) \sum_{j=-k}^{k-i} q_j q_{j-i}} \tag{29}$$

is an estimator of $q\left(1\right)^2 = 1$, and hence (again with appropriate choice of bandwidth and kernel), will tend to one.

Table 1 also shows computed the shift and scale terms (28) and (29), respectively, for the Bartlett kernel

$$w(i,m) = 1 - \frac{i}{(m+1)}, \qquad i = 1, 2, ..., m, \tag{30}$$

for different values of m, again when the quarterly X-11 filter is applied. These confirm that the terms do have effects that disappear for large values of m. Nevertheless, it is an empirical matter whether the bandwidths typically used are sufficiently wide to capture the high order MAs resulting from seasonal adjustment in practice. The Monte Carlo analysis of the next section sheds light on this question.

3.5 Variance ratio and M type tests

Finally we examine the asymptotic distributions of the variance ratio test (VRT) of Breitung (2002), given by

$$VRT = T^{-2} \frac{\sum_{t=1}^{T} \left(\sum_{j=1}^{t} y_j\right)^2}{\sum_{t=1}^{T} y_t^2} \tag{31}$$

together with tests proposed by Stock (1999) and popularized by Perron and Ng (1996), which are

$$MSB = \left(T^{-2} \frac{\sum_{t=1}^{T} y_{t-1}^2}{s_l^2} \right)^{1/2} \tag{32}$$

$$MZ_\alpha = \frac{T^{-1} y_T^2 - s_l^2}{2T^{-2} \sum y_{t-1}^2} \tag{33}$$

where s_l^2 is an estimator of the long run variance, given by (25) in the preceding subsection. The unit root test statistic given by

$$MZ_t = MSB \times MZ_\alpha \tag{34}$$

proposed by Perron and Ng (1996) is also considered. The following proposition gives the asymptotic distributions for all these statistics for the near-integrated DGP.

Proposition 4 *For an unadjusted series following the near integrated process of (9), the tests statistics of (31), (32), (33) and (34) applied to the seasonally adjusted series of (10), have asymptotic distributions*

$$VRT \Rightarrow \frac{\int_0^1 \left[\int_0^r J_c\left(\lambda, g\right) dg\right]^2 dr}{\int_0^1 J_c\left(\lambda, r\right)^2 dr} \tag{35}$$

$$MSB \Rightarrow \left(\frac{\int_0^1 J_c(\lambda, r)^2 \, dr}{\varpi^2} \right)^{1/2}. \tag{36}$$

$$MZ_\alpha \Rightarrow \frac{\left(J_c(\lambda, r)^2 - \varpi^2 \right)}{2 \int_0^1 J_c(\lambda, r)^2 \, dr} \tag{37}$$

$$MZ_t \Rightarrow \frac{\left(J_c(\lambda, r)^2 - \varpi^2 \right)}{2 \sqrt{\int_0^1 J_c(\lambda, r)^2 \, dr}} \tag{38}$$

where ϖ^2 *is defined in (29).*

Note that the distribution of the VRT test in (35) is free from nuisance parameters and hence is unaffected by seasonal adjustment; this is also the case for the MSB, MZ_α and MZ_t tests, provided that ϖ^2 satisfactorily estimates $q(1)^2 = 1$. The asymptotic properties of ϖ for the Bartlett kernel as the bandwidth m increases are investigated in Table 1, labelled there as the PP scale factor.

3.6 Mean effects and detrending

Purely for simplicity of exposition, the discussion above does not allow for deterministic terms. However, results collected in propositions 1 to 4 extend in a straightforward way to deterministic components. In conjunction with the zero mean stochastic process (9), assume the observed unadjusted series is given by

$$x_t^u = y_t^u + \mu_t^u \tag{39}$$

where μ_t^u is the deterministic component, with $\mu_t^u = \sum_{j=1}^s \delta_{jt} \beta_s + \kappa t$ where δ_{jt} is a zero-one dummy variable for season j and κ is the trend coefficient. As noted in subsection 2.1, the seasonal unit roots embedded in the X-11 filter annihilate seasonal intercept effects, while Bell (2012) shows that the symmetric filter reproduces linear (and, indeed, higher order) trends[2]. Therefore, after application of the symmetric X-11 filter, the adjusted series corresponding to (39) is $x_t^f = y_t^f + \mu_t^f$ where $\mu_t^f = \delta + \kappa t$ and δ is constant over seasons.

Application of unit root tests with Ordinary Least Squares (OLS) detrending leads to analogous results to those of Propositions 1 to 4 above, but replacing the Ornstein-Uhlenbeck and Brownian motion processes by demeaned and detrended Ornstein-Uhlenbeck and Brownian motion processes, respectively. If only mean effects are relevant and no trends are allowed for in the unit root tests, then the corresponding demeaned processes apply.

In addition, pseudo-Generalized Least Squares (GLS) detrending can be applied, in which case the statistics are computed using the local GLS detrended values $\widetilde{y}_t^f$; see Elliott, Rothenberg and Stock (1996, pp. 824-825) for details. However, in this case the term $-T^{-1}\widetilde{y}_1^f$ needs to be added to the numerator of MZ_α in (33). When only an intercept δ it is considered the distributions of the analyzed tests coincide with those reported in Propositions 1 to 4, while with an intercept and trend we have analogous results to those of Propositions 1 to 4 but with the Ornstein-Uhlenbeck and Brownian motion

[2] The X-11 filter does not, however, entirely eliminate seasonal trends. This would requires two applications of the seasonal summation filter $S(L) = 1 + L + ... + L^{s-1}$. While the symmetric X-11 filter is close to containing $[S(L)]^2$, it does not do so (Bell, 2012).

processes replaced by the corresponding detrended processes; see Elliott, Rothenberg and Stock (1996, pp. 824-825) and Theorem 1 in Ng and Perron (1999) for details.

In the context of the Dickey and Fuller test two further alternative methods of dealing with the deterministic part have been considered, the Recursive Mean Adjustment (see Taylor (2002) for details) and the Weighted Symmetric Least Squares (WSLS) (see Pantula, Gonzalez-Farias and Fuller (1994), Park and Fuller (1995) and Fuller (1996) for details). Note that both methods deliver better results in terms of power than pseudo-GLS detrending in the unconditional as shown by Pantula, Gonzalez-Farias and Fuller (1994) (figure 2) and Park and Fuller (1995) in the case of WSLS and by Taylor (2002) table 4.1 in the case of Recursive Mean Adjustment. For Recursive Mean Adjustment, the distribution of the DF/ADF test will follow the lines of Proposition 2 but replacing the Ornstein-Uhlenbeck and Brownian motion processes by the appropriated processes, as in (4.1) of Theorem 4.1 and (4.8) of Theorem 4.2 in Taylor (2002). Finally in the case of WSLS the results of Proposition 2 will need to be adapted to the expression presented in Theorems 10.1.7 and 10.1.8 of Fuller (1996); see also (4.8) of Theorem 4.2 and remarks 4.4 and 4.10 in Rodrigues and Taylor (2004).

4 Size and Power Analysis

4.1 Asymptotic power functions

Figures 1 to 9 show the asymptotic power functions of the ADF, PP, $Z(t_\alpha)$, VRT, MSB, $MZ_\alpha$and MZ_t tests for unfiltered and filtered data generated using (9), with a nominal test size of 0.05. The results are obtained in an equivalent way to those reported in Figure 1 in Elliot, Rothemberg and Stock (1996), with the asymptotic distributions approximated by discrete realizations for a sample size of $T = 400$ and based on $20,000$ replications. Note that although the DGPs are initialized at zero, we generate an extra 100 observations at the beginning so that the starting value effectively satisfies the unconditional assumption, with additional values also generated at the end in order to be able to employ the two-sided symmetric X-11 filter over the central 400 values for both the filtered and the filtered data. Quarterly X-11 weights are employed and all results are computed for models that include an intercept.

For the ADF, MSB, $MZ_\alpha$and MZ_t tests, we consider both OLS and GLS detrending. However, only OLS detrending is considered for the PP, $Z(t_\alpha)$ and VRT tests. For the PP test only OLS detrending has been proposed, while Breitung and Taylor (2003) show that GLS detrending does not improve the performance of the VRT test in comparison with OLS detrending. In addition, we also consider Recursive Mean Adjustment and $WSLS$ detrending for the ADF test.

The Propositions of Section 3 suggest that the choice of augmentation lag or bandwidth parameter is important for the asymptotic power functions of unit root tests applied to seasonally adjusted data. For the ADF test of (15) we therefore investigate four automatic rules to determine the order of augmentation that would be applied for a sample size of $T = 400$. These set p: to $\left[T^{1/2}\right]$, labeled as $k2$ and resulting in $p = 20$, $\left[T^{1/3}\right]$ labeled as $k3$ ($p = 7$), $\ell 4 = int\left(4\left[\frac{T}{100}\right]^{1/4}\right)$ labelled as $l4$ ($p = 5$) and finally $\ell 12 = int\left(12\left[\frac{T}{100}\right]^{1/4}\right)$ labelled as $l12$ (yielding $p = 16$).

For the PP, MSB, $MZ_\alpha$and MZ_t tests, we use the Bartlett and Quadratic spectral kernels. Here we follow the approach of Newey and West (1994, equations (3.8) to (3.15) and Table 1) for selection of the bandwidth parameter employed in conjunction with two kernels. In an obvious notation, we use B and QS in the figures to denote the Bartlett and Quadratic spectral kernels, respectively. For the Bartlett kernel, the bandwidth is specified as $m = \widehat{\delta}T^{1/(2q+1)}$ in which $\widehat{\delta} = c_\delta \left(\hat{s}^{(q)}/\hat{s}^{(0)}\right)^{2/(2q+1)}$, $c_\delta = 1.1447$, $q = 1$ and

$$\hat{s}^{(q)} = 2\sum_{j=1}^{n} j^q \widehat{\gamma}_j, \quad \hat{s}^{(0)} = \widehat{\gamma}_0 + 2\sum_{j=1}^{n} \widehat{\gamma}_j, \quad \widehat{\gamma}_j = T^{-1}\sum_{t=i+1}^{T} \hat{u}_t \hat{u}_{t-j}$$

$n = \left[12\left(T/100\right)^{2/9}\right]$. The quadratic spectral kernel has $m = T - 1$ and

$$w(i,m) = \frac{25}{12\pi^2 x_i^2}\left(\frac{\sin\left(6\pi x_i/5\right)}{6\pi x_i/5} - \cos\left(6\pi x_i/5\right)\right) \tag{40}$$

with $x_i = i/\left(\widehat{\delta}T^{1/(2q+1)}\right)$, where $\widehat{\delta}$ is defined as above but here $q = 2$, $c_\delta = 1.3221$ and $n = \left[4\left(T/100\right)^{2/25}\right]$. The choices made in both cases reflect the recommendations of Newey and West (1994). For the purposes of computing the asymptotic power functions we again employ values corresponding to $T = 400$.

Figure 1 illustrates a number of features of interest for the ADF test. Although the use of augmentation generally delivers good asymptotic size, the $k3$ rule with $p = 7$ delivers an over-sized test. In other words, this is not a sufficient order of AR augmentation to asymptotically take account of the MA features of the seasonally adjusted data. On the other hand, the $l4$ rule with the lower order $p = 5$ has relatively good asymptotic size. Therefore, increasing the order of augmentation does not necessarily improve size, due to the complicated scale and shift effects in (19), illustrated in Table 1. However, high levels of augmentation such as 16 (indicated by $l12$) or 20 ($k2$) yield relatively good asymptotic size, albeit at the cost of power substantially lower than that of the DF test applied to unadjusted data as c moves away from zero. There is nevertheless a nontrivial loss of power for the well-sized l4 augmentation rule also, especially around $c = 24$.

As pointed out by Pantula, Gonzalez-Farias and Fuller (1994) and Elliot (1999), the ADF test with OLS detrending outperforms the ADF with GLS detrending for some values of c, as can be observed if we compare Figures 1 and 2 for the unfiltered data (that is, the lines labelled DF). In general, the ADF power functions for seasonally adjusted data have similar relationships to the DF one in Figure 2 as in Figure 1, although the undersizing of the ADF test with high augmentation orders is more marked in Figure 2. This is again a consequence of the effects seen in Table 1, where the distribution of the ADF statistic for filtered data is shifted to the right for the corresponding values of $p = 16$ and 20, and hence we anticipate undersizing and lower power than for the unfiltered data under local alteratives as also observed to a lesser extent in Figure 1. The same comment applies for the $l4$ case, but (as indicated by the values of Table 1) the scale and shift distortions are smaller. GLS detrending in Figure 2 also increases the oversizing of the test with $p = 7$ (denoted $l4$) compared with OLS detrending in Figure 1. These patterns in relation to the DF case are maintained in Figures 3 and 4. However, Recursive Mean Adjustment detrending in Figure 3 effectively shifts all the power functions upwards, so that the DF case is also oversized, while $WSLS$ detrending in Figure 4 yields similar results to GLS detrending.

In the case of the VRT test we clearly observe that the performance of the test in terms of size and power is observationally equivalent with unfiltered and filtered data, where the latter are indicated using the suffix F in Figure 5; this is anticipated from Proposition 4. As implied by the PP shift and scale terms in Table 1, the power function of this test with filtered data is a little below the one obtained with unfiltered data in

Figure 5. This feature also applies for the MSB test in Figures 6 and 7, which correspond to OLS and GLS detrending, respectively.

Finally in the case of the $MZ_\alpha$and MZ_t tests (see Figures 8 and 9), the power function for filtered data is below the power function with unfiltered data. Hence we clearly observe here that ϖ^2 does not satisfactorily approach $q(1)^2$ when the above choices are made for the window parameters.

4.2 Monte Carlo analysis

Tables 2 to 8 collect results for the empirical size and power for $c = 0,\ 1,\ 2, 5$ and 10 using $N = 400$ observations. Rather than the uncorrelated disturbance assumption of (9), the DGP for the unfiltered data in this case has a nonseasonal or seasonal MA form as follows:

$$\begin{aligned} y_t^u &= \exp\left(\frac{c}{T}\right) y_{t-1}^u + \epsilon_t \qquad (41) \\ \epsilon_t &= \varepsilon_t - \theta\varepsilon_{t-1} \quad \theta = \pm.5 \\ \epsilon_t &= \varepsilon_t - \Theta\varepsilon_{t-4} \quad \Theta = .5 \\ & \varepsilon_t \sim NIID\,(0, 1)\,. \end{aligned}$$

In tables 2 to 5, where ADF test results are collected, the order of augmentation for the unfiltered data is determined with the rules $k2$, $k3$, $l4$ and $l12$ and these are used also for the filtered data. Across all these tables, the values for filtered data in the columns $k2\_f$, $l4\_f$ and $l12\_f$ are always smaller than the corresponding ones obtained for unfiltered data in the columns headed $k2$, $l4$ and $l12$. However, using the rule $k3$, the filtered values ($k3\_f$ column) are larger than the unfiltered ones ($k3$ column). Hence these empirical results mimic those reported for a random walk in Figures 1 to 4. The results also emphasize the size distortions that can result from applying the ADF test to seasonally adjusted data. For example, with OLS detrending in Table 2 and applying the relatively generous $k2$ rule, the empirical size is only 0.03 rather than the nominal 0.05. On the other hand, the more parsiminous $k3$ and $l4$ rules are badly over-sized in the presence of a negative seasonal MA coefficient across all four tables. Although the results for the seasonally unadjusted case are in line with the findings of Schwert (1989), it is clear that the X-11 seasonal adjustment filter does effectively nothing to help (indeed, size is worse in each case for $k3\_f$ than for $k3$) unless a very large augmentation order is employed.

In the case of the VRT test, the results in Table 6 (as expected) have empirical size and power that are very similar for the unfiltered and filtered data. For the PP test in Table 6 and the M-type tests in Tables 7 and 8, we clearly observe that the values obtained for the filtered data are often smaller than those for the unfiltered data. In this sense, the results for Tables 6-8 show corresponding behaviour to that observed in Figures 5-7. However, the sizes of these tests are often very poor. For example, the PP test has over 50 percent rejections of the null hypothesis when applied to the seasonal MA process with $\Theta = 0.5$. It appears that the nature of these MA processes is not well accounted for by either the Bartlett or quadratic spectral window when applied with the values recommended by Newey and West (1994).

5 Conclusions

This paper has demonstrated, both analytically and through Monte Carlo simulations, the implications of seasonal adjustment for the properties of zero frequency unit root tests applied to integrated and near-integrated processes. Although previous contributions relating to the ADF test suggest that the usual asymptotic results continue to apply, these examine only the null distribution and rest on an invertibility assumption that is unlikely to be satisfied by data after application of the commonly used X-11 seasonal adjustment filters.

In one sense, our analysis is reassuring, since we show that the invertibility assumption is not crucial in that the use of a sufficiently high order of augmentation does, indeed, deliver the usual ADF asymptotic distributions. However, the order of augmentation required can be very large, due to both non-invertibility and the length of the two-sided filter used in adjustment. Further, the high orders required to deliver good size lead to substantial power losses for adjusted data compared with direct testing on the unadjusted series.

We also investigate the properties of the PP, variance ratio and M-type tests. Although the variance ratio tests are robust to the use of adjusted data, the other tests examined may not be. More specifically, the PP and M-type tests require consistent estimation of the long-run variance, with consistency requiring the kernel employed to take account of the long MA component arising from the use of the X-11 seasonal adjustment filter.

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6 Appendix

Derivation of (11):

The filtered near-integrated process of (10) is given by

$$
\begin{aligned}
y_t^f &= \varphi_T^{t-1} y_1^f + \sum_{j=0}^{t-2} \varphi_T^j u_{t-j} = \varphi_T^{t-1} y_1^f + \sum_{j=0}^{t-2} \varphi_T^j \sum_{i=-k}^{k} q_i \epsilon_{t-j+i} \\
&= \varphi_T^{t-1} y_1^f + q_k \epsilon_{t+k} + (q_k \varphi_T + q_{k-1})\epsilon_{t+k-1} + (q_k \varphi_T^2 + q_{k-1}\varphi_T + q_{k-2})\epsilon_{t+k-2} + \cdots \\
&\quad + (q_k \varphi_T^{k-1} + \cdots + q_1)\epsilon_{t+1} + (q_k \varphi_T^k + ... + q_1 \varphi_T + q_0)\epsilon_t \\
&\quad + (q_k \varphi_T^{k+1} + \cdots + q_0 \varphi_T + q_1)\epsilon_{t-1} + \cdots + (q_k \varphi_T^{2k} + \cdots + q_0 \varphi_T^k + \cdots + q_k)\epsilon_{t-k} + \cdots \\
&\quad + (q_k \varphi_T^{t-2} + q_{k-1}\varphi_T^{t-3} + \cdots + q_k \varphi_T^{t-2k-1})\epsilon_{k+1} + ... + (q_1 \varphi_T^{t-2} + \cdots + q_k \varphi_T^{t-k-5})\epsilon_1 + ... \\
&\quad + (q_{k-2}\varphi_T^{t-2} + q_{k-1}\varphi_T^{t-3} + q_k \varphi_T^{t-4})\epsilon_{4-k} + (q_{k-1}\varphi_T^{t-2} + q_k \varphi_T^{t-3})\epsilon_{3-k} + q_k \varphi_T^{t-2}\epsilon_{2-k} \\
&= \varphi_T^{t-1} y_1^f + \sum_{j=0}^{t-2} \epsilon_{t-j} \sum_{i=-k}^{k} q_i \varphi_T^{j-i} + \sum_{j=1}^{k} \epsilon_{t+j} \sum_{i=j}^{k} q_i \varphi_T^{i-j} + \sum_{j=1}^{k} \epsilon_{2-j} \sum_{i=j}^{k} q_i \varphi_T^{t-2+k-i} \\
&\quad - \sum_{j=0}^{k-1} \epsilon_{t-j} \sum_{i=j+1}^{k} q_i \varphi_T^{j-i} - \sum_{j=0}^{k-1} \epsilon_{2+j} \sum_{i=j+1}^{k} q_i \varphi_T^{t-2+i}. \qquad (42)
\end{aligned}
$$

Also note that, in the last expression above,

$$\sum_{i=-k}^{k} q_i \varphi_T^{j-i} = \varphi_T^j q(\varphi_T)$$

where $q(\varphi_T)$ is the polynomial associated with the symmetric X-11 filter $q(L)$ with $L = \varphi_T$.

Proof of Proposition 1

Following the lines of the A.1 preliminaries in the appendix of Canjels and Watson (1997), let $v_t = \sum_{j=0}^{t-2} \epsilon_{t-j} \sum_{i=-k}^{k} q_i \varphi_T^{j-i}$ with $\varphi_T = 1 + c/T$. Then $T^{-1/2} v_{[rt]} \Rightarrow \sigma q(1) J_c(r)$ where $J_c(r)$ denotes a diffusion process generated by $J_c(r) = W(r) + c\int_0^r e^{(r-g)c} W(g)\,dg$ where $W(r)$ is a standard Brownian motion process $T^{-1/2} \sum_{t=1}^{[rT]} \epsilon_t \Rightarrow \sigma W(r)$ (see Phillips (1987) for details). Similarly $T^{-1/2} y_1^f = \sum_{i=0}^{[\lambda T]} \varphi_T^i u_{1-i} \Rightarrow \sigma q(1) \bar{J}_c(\lambda)$ where $\bar{J}_c(\lambda)$ denotes the diffusion process $\bar{J}_c(\lambda) = \bar{W}(\lambda) + c\int_0^\lambda e^{(\lambda-g)c} \bar{W}(g)\,dg$ with $\bar{W}(\lambda)$ been a standard Brownian motion process $T^{-1/2} \sum_{t=-[\lambda T]+k}^{1} \epsilon_t \Rightarrow \sigma \bar{W}(\lambda)$. Finally note that:

$$w_t = v_t + \varphi_T^{t-1} y_1^f \qquad T^{-1/2} w_{[rT]} \Rightarrow q(1)\left[J_c(r) + e^{rc}\bar{J}_c(\lambda)\right] := \sigma q(1) J_c(\lambda, r) \qquad (43)$$

Using (42), it can be seen that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T} y_{t-1}^{f}\epsilon_t = T^{-1}\sum_{t=1}^{T}&\left(\varphi_T^{t-1}y_1^f+\sum_{j=0}^{t-1}q\left(\varphi_T^{-j}\right)\epsilon_{t-j-1}+\sum_{j=1}^{k}\epsilon_{t+j-1}\sum_{i=j}^{k}q_i\varphi_T^{i-j}\right.\\
&\left.+\sum_{j=1}^{k}\epsilon_{2-j-1}\sum_{i=j}^{k}q_i\varphi_T^{t-2+k-i}-\sum_{j=0}^{k-1}\epsilon_{t-j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{j-i}-\sum_{j=0}^{k-1}\epsilon_{2+j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{t-2+i}\right)\epsilon_t\\
&=T^{-1}\sum_{t=1}^{T}\left(\varphi_T^{t-1}y_1^f+\sum_{j=0}^{t-1}q\left(\varphi_T^{-j}\right)\epsilon_{t-j-1}\right)\epsilon_t+T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\epsilon_{t+j-1}\sum_{i=j}^{k}q_i\varphi_T^{j-i}\right)\epsilon_t\\
&+T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\epsilon_{2-j-1}\sum_{i=j}^{k}q_i\varphi_T^{t-2+k-i}\right)\epsilon_t-T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\epsilon_{t-j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{j-i}\right)\epsilon_t\\
&-T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\epsilon_{2+j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{t-2+i}\right)\epsilon_t. \qquad (44)
\end{aligned}
$$

Note that based on (43) and the CMT it is possible to write

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(\varphi_T^{t-1}y_1^f+\sum_{j=0}^{t-1}q\left(\varphi_T^{-j}\right)\epsilon_{t-j-1}\right)\epsilon_t &= T^{-1}\sum_{t=1}^{T}w_{t-1}\epsilon_t\\
\Rightarrow \sigma^2 q\left(1\right)\int_0^1 J_c\left(\lambda,r\right)dW(r) &= \sigma^2\int_0^1 J_c\left(\lambda,r\right)dW(r)
\end{aligned}
$$

since the weights of the symmetric X-11 filter sum to unity, and where $J_c\left(\lambda,r\right)$ is an Ornstein-Uhlenbeck process defined in (43) and $W\left(r\right)$ is an standard Brownian motion. Further, due to the *iid* behavior of ϵ_t, it is straightforward to see that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\epsilon_{t+j-1}\sum_{i=j}^{k}q_i\varphi_T^{j-i}\right)\epsilon_t &= T^{-1}\sum_{t=1}^{T}\left(\epsilon_t^2\sum_{i=j}^{k}q_i\varphi_T^{j-i}+\sum_{j=2}^{k}\epsilon_{t+j-1}\epsilon_t\sum_{i=j}^{k}q_i\varphi_T^{j-i}\right)\\
&\Rightarrow \sigma^2\sum_{i=1}^{k}q_i
\end{aligned}
$$

note that $\varphi_T \simeq 1+c/T$ hence as $T\to\infty$ we have that $\varphi_T\to 1$. Also

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\epsilon_{t-j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{i-j}\right)\epsilon_t &\to 0 \qquad T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{k}\epsilon_{2-j-1}\sum_{i=j}^{k}q_i\varphi_T^{t-2+k-i}\right)\epsilon_t\to 0\\
T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\epsilon_{t-j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{j-i}\right)\epsilon_t &\to 0 \qquad T^{-1}\sum_{t=1}^{T}\left(\sum_{j=0}^{k-1}\epsilon_{2+j-1}\sum_{i=j+1}^{k}q_i\varphi_T^{t-2+i}\right)\epsilon_t\to 0
\end{aligned}
$$

Hence the asymptotic distribution corresponding to (44) is

$$
T^{-1}\sum_{t=1}^{T}y_{t-1}^{f}\epsilon_t \Rightarrow \sigma^2\int_0^1 J_c\left(\lambda,r\right)dW(r)+\sigma^2\sum_{i=1}^{k}q_i \qquad (45)
$$

To obtain the asymptotic distribution of the normalized bias $T(\widehat{\varphi}_T - \varphi_T)$ for the filtered for the near integrated process, note that

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} u_t &= T^{-1}\sum_{t=1}^{T} y_{t-1}^{f}\left(q_k\epsilon_{t+k}+\cdots+q_0\epsilon_t+\cdots+q_k\epsilon_{t-k}\right)\\
&= T^{-1}\sum_{t=1}^{T}\Big\{\left(q_k y_{t+k-1}^{f}\epsilon_{t+k}+\cdots+q_0 y_{t-1}^{f}\epsilon_t+\cdots+q_k y_{t-k-1}^{f}\epsilon_{t-k}\right)\\
&\quad -\sum_{i=1}^{k} q_i\left(y_{t+i-1}^{f}-y_{t-1}^{f}\right)\epsilon_{t+i}+\sum_{i=1}^{k} q_i\left(y_{t-1}^{f}-y_{t-i-1}^{f}\right)\epsilon_{t-i}\Big\}
\end{aligned}
\tag{46}
$$

Now

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(y_{t+i-1}^{f}-y_{t-1}^{f}\right)\epsilon_{t+i} &= T^{-1}\sum_{t=1}^{T}\Big\{\left(1-\varphi_T L\right)y_{t+i-1}^{f}+\left(1-\varphi_T L\right)y_{t+i-2}^{f}+\\
&\qquad \cdots+\left(1-\varphi_T L\right)y_{t}^{f}-\frac{c}{T}\sum_{j=1}^{i} y_{t+i-1-j}^{f}\Big\}\epsilon_{t+i}\\
&= T^{-1}\sum_{t=1'}^{T}\{[q\left(L\right)\epsilon_{t+i-1}]\epsilon_{t+i}+[q\left(L\right)\epsilon_{t+i-2}]\epsilon_{t+i}+\\
&\qquad \cdots+[q\left(L\right)\epsilon_{t}]\epsilon_{t+i}\}+o_p\left(1\right)\\
&\to \sigma^2\sum_{j=1}^{i} q_j
\end{aligned}
\tag{47}
$$

as $T^{-2}c\sum_{t=1}^{T}\sum_{j=1}^{i} y_{t+i-1-j}^{f}\epsilon_{t+i} = o_p\left(1\right)$ and, similarly,

$$
\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(y_{t-1}^{f}-y_{t-i-1}^{f}\right)\epsilon_{t-i} &= T^{-1}\sum_{t=1}^{T}\Big(\left(1-\varphi_T L\right)y_{t-1}^{f}+\left(1-\varphi_T L\right)y_{t-2}^{f}+\\
&\qquad \cdots+\left(1-\varphi_T L\right)y_{t-i}^{f}-\frac{c}{T}\sum_{j=1}^{i} y_{t-1-j}^{f}\Big)\epsilon_{t-i}\\
&\to \sigma^2\sum_{j=0}^{i-1} q_j
\end{aligned}
\tag{48}
$$

Further,

$$
\begin{aligned}
\sum_{i=1}^{k} q_i-\sum_{i=1}^{k} q_i\sum_{j=1}^{i} q_j+\sum_{i=1}^{k} q_i\sum_{j=0}^{i-1} q_j &= \sum_{i=1}^{k} q_i(1+q_0-q_i)\\
&= \frac{1}{2}(1-q_0)(1+q_0)-\sum_{i=1}^{k} q_i^2\\
&= \frac{1}{2}\left[1-q_0^2-2\sum_{i=1}^{k} q_i^2\right]=\frac{1}{2}\left[1-\sum_{i=-k}^{k} q_i^2\right]
\end{aligned}
\tag{49}
$$

where we use the symmetry of $q(L)$ and also the relationship $\sum_{i=1}^{k} q_i = \frac{1}{2}[1-q_0]$ which follows from symmetry together with $q(1)=1$. Therefore, using $q(1)=1$, (45) to (49) satisfies

$$T^{-1}\sum_{t=1}^{T} y_{t-1}^{f} u_t \Rightarrow \sigma^2 \int_0^1 J_c(\lambda, r)\, dW(r) + \frac{1}{2}\sigma^2 \left[1 - \sum_{i=-k}^{k} q_i^2\right]. \tag{50}$$

The denominator of (12) follows from (43) and the CMT, as

$$T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^2 \Rightarrow \sigma^2 q(1)^2 \int^2 J_c(s,r)^2\, dr = \sigma^2 \int J_c(s,r)^2\, dr \tag{51}$$

Using (49) and (51), together with $\varphi_T = 1 - \frac{c}{T}$, then yields (13).

The t-ratio for the filtered data is

$$t_{(\hat{\varphi}_T - \varphi_T)} = T(\hat{\varphi}_T - \varphi_T) \times \frac{\sqrt{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^2}}{\sqrt{T^{-1}\sum_{t=1}^{T}\left(y_t^f - \hat{\varphi}_T y_{t-1}^{f}\right)^2}}.$$

Since

$$T^{-1}\sum_{t=1}^{T}\left(y_t^f - \hat{\varphi}_T y_{t-1}^{f}\right)^2 = T^{-1}\sum_{t=1}^{T} u_t^2 + o_p(1) \Rightarrow \sigma^2 \sum_{i=-k}^{k} q_i^2$$

and using (49) and (51), (14) is easily obtained.

Proof of (17)

With augmentation of the test regression applied to seasonally adjusted data, adding and subtracting $\sum_{i=1}^{p}\phi_i^p\left(\Delta - \frac{c}{T}L\right)y_{t-i}^f = \sum_{i=1}^{p}\phi_i^p u_{t-i}$ yields

$$\begin{aligned}
\Delta y_t^f &= \frac{c}{T} y_{t-1}^f + u_t \\
&= \frac{c}{T}\left(1 - \phi_1^p L - \cdots - \phi_p^p L^p\right) y_{t-1}^f + \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i}^f + u_t - \sum_{i=1}^{p}\phi_i^p\left(\Delta - \frac{c}{T}L\right) y_{t-i}^f \\
&= \frac{c}{T}\phi^p(L) y_{t-1}^f + \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i}^f + \phi^p(L) u_t \\
&= \frac{c}{T}\phi^p(1) y_{t-1}^f + \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i}^f + \phi^p(L) u_t + \frac{c}{T}\sum_{i=1}^{p}\phi_i^p (1 - L^p) y_{t-1}^f \\
&= \frac{c}{T}\phi^p(1) y_{t-1}^f + \sum_{i=1}^{p}\phi_i^p \Delta y_{t-i}^f + e_t^p + o_p(1)
\end{aligned}$$

where $\phi^p(L) = 1 - \phi_1^p L - \ldots - \phi_p^p L^p$ and $e_t^p = \phi^p(L) u_t$. Note that $(1 - L^p) y_{t-1}^f = \Delta y_{t-1}^f + \cdots + \Delta y_{t-p+1}^f$. Analogously to Rodrigues and Taylor (2004, Proposition 2.1), it is possible to see that $\alpha = \frac{c}{T}\phi^p(1) + O\left(T^{-2}\right)$ in (15) and that $e_t^p = u_t - \phi_1^p \Delta y_{t-1}^f - \cdots - \phi_p^p \Delta y_{t-p}^f + \frac{pc}{T} y_{t-1}^f = \left(1 - \phi_1^p L - \cdots - \phi_p^p L^{p+1}\right) u_t + o_p(1)$. Hence, we have

$$e_t^p = \phi^p(L) q(L) \epsilon_t = \theta^p(L) \epsilon_t$$

where $\theta^p(L)$ is an asymmetric two-sided moving average, with $k+p$ nonzero lags and k nonzero leads; this establishes (17) of the text.

Proof of Proposition 2

When the ADF regression for the seasonally adjusted near integrated process is augmented to order p, OLS estimation yields

$$\begin{bmatrix} \hat{\alpha}-\alpha \\ \hat{\phi}_1-\phi_1^p \\ \vdots \\ \hat{\phi}_p-\phi_p^p \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^T \left(y_{t-1}^f\right)^2 & \sum_{t=1}^T y_{t-1}^f \Delta y_{t-1}^f & \cdots & \sum_{t=1}^T y_{t-1}^f \Delta y_{t-p}^f \\ \sum_{t=1}^T y_{t-1}^f \Delta y_{t-1}^f & \sum_{t=1}^T \left(\Delta y_{t-1}^f\right)^2 & \cdots & \sum_{t=1}^T \Delta y_{t-1}^f \Delta y_{t-p}^f \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^T y_{t-1}^f \Delta y_{t-p}^f & \sum_{t=1}^T \Delta y_{t-1}^f \Delta y_{t-p}^f & \cdots & \sum_{t=1}^T \left(\Delta y_{t-p}^f\right)^2 \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} \sum_{t=1}^T e_t^p y_{t-1}^f \\ \sum_{t=1}^T e_t^p \Delta y_{t-1}^f \\ \vdots \\ \sum_{t=1}^T e_t^p \Delta y_{t-p}^f \end{bmatrix}$$

The different rates of convergence that apply to the coefficients corresponding to the regressors leads us to consider

$$\begin{bmatrix} T\left(\hat{\alpha}-\alpha\right) \\ T^{\frac{1}{2}}\left(\hat{\phi}-\phi^p\right) \end{bmatrix} = \begin{bmatrix} T^{-2}\sum_{t=1}^T \left(y_{t-1}^f\right)^2 & h' \\ h & \hat{\Gamma} \end{bmatrix}^{-1} \times \begin{bmatrix} T^{-1}\sum_{t=1}^T e_t^p y_{t-1}^f \\ \hat{b} \end{bmatrix} \qquad (52)$$

where the $(p\times 1)$ vector h has i^{th} element $T^{-\frac{3}{2}}\sum_{t=1}^T y_{t-1}^f \Delta y_{t-i}^f \to 0$ and the $(p\times 1)$ vector $\hat{b}$ has i^{th} element $\sum_{t=1}^T e_t^p \Delta y_{t-i}^f$, while $\hat{\Gamma}$ is the estimated covariance matrix to order p for Δy_t^f.

The asymptotic distribution of $T^{-2}\sum_{t=1}^T \left(y_{t-1}^f\right)^2$ remains unchanged from (51), so that to obtain the distribution of $T\left(\hat{\alpha}-\alpha\right)$ we need to consider only

$$T^{-1}\sum_{t=1}^T y_{t-1}^f e_t^p = T^{-1}\sum_{t=1}^T y_{t-1}^f \left(\theta_{-k}^p \epsilon_{t+k} + \cdots + \theta_{-1}^p \epsilon_{t+1} + \theta_0^p \epsilon_t + \theta_1^p \epsilon_{t-1} + \cdots + \theta_{k+p}^p \epsilon_{t-k-p}\right)$$

$$= T^{-1}\sum_{t=1}^T \Bigg\{ \left(\theta_{-k}^p y_{t+k-1}^f \epsilon_{t+k} + \cdots + \theta_{-1}^p y_t^f \epsilon_{t+1} + \theta_0^p y_{t-1}^f \epsilon_t + \cdots + \theta_{k+p}^p y_{t-k-p-1}^f \epsilon_{t-k-p}\right)$$

$$- \sum_{i=1}^k \theta_{-i}^p \left(y_{t+i-1}^f - y_{t-1}^f\right)\epsilon_{t+i} + \sum_{i=1}^{k+p} \theta_i^p \left(y_{t-1}^f - y_{t-i-1}^f\right)\epsilon_{t-i} \Bigg\}$$

Using (45), together with (47) and (48), we have

$$T^{-1}\sum_{t=1}^T y_{t-1}^f e_t^p \Rightarrow \frac{1}{2}\sigma^2\theta^p(1)\int J_c(s,r)\,dW(r) + \theta^p(1)\,\sigma^2\sum_{i=1}^k q_i - \sigma^2\sum_{i=1}^k \theta_{-i}^p \sum_{j=1}^i q_j \qquad (53)$$

$$+\sigma^2\sum_{i=1}^{k+1}\theta_i^p\sum_{j=0}^{i-1} q_j + \sigma^2\sum_{i=k+2}^{k+p}\theta_i^p\sum_{j=0}^{k} q_j.$$

This expression, together with (51), yields the asymptotic distribution given in (18).
The corresponding t-ratio is given by

$$t_{(\hat{\alpha}-\alpha)} = T\left(\hat{\alpha}-\alpha\right) \times \frac{\sqrt{T^{-2}\sum_{t=1}^{T}\left(y_{t-1}^{f}\right)^{2}}}{\sqrt{T^{-1}\sum_{t=1}^{T}\left(\Delta y_{t}^{f}-\hat{\alpha}y_{t-1}^{f}-\sum_{j=1}^{k}\hat{\phi}_{i}\Delta y_{t-j}^{f}\right)^{2}}}$$

and (19) is obtained by noting that form (18) we have $T\left(\hat{\alpha}-\alpha\right) = O_p\left(1\right)$ hence $\left(\hat{\alpha}-\alpha\right) = o_p\left(1\right)$ and then $\hat{\alpha} \to \alpha$and that from (52) it is also possible to show that $\hat{\phi}_i \to \phi_i^p$, then

$$t_{\hat{\alpha}} \Rightarrow \frac{\sigma^2\theta^p\left(1\right)\int J_c\left(s,r\right)dW(r) + \sigma^2\left(\theta^p\left(1\right)\sum_{i=1}^{k}q_i - \sum_{i=1}^{k}\theta_{-i}^{p}\sum_{j=1}^{i}q_j + \sum_{i=1}^{k+1}\theta_i^p\sum_{j=0}^{i-1}q_j + \sum_{i=k+2}^{k+p}\theta_i^p\sum_{j=0}^{k}q_j\right)}{\sqrt{\sigma^2\int J_c\left(s,r\right)^2 dr}\sqrt{Var(e_t^p)}}$$

$$= \frac{\theta^p\left(1\right)\int J_c\left(s,r\right)dW(r) + \left(\theta^p\left(1\right)\sum_{i=1}^{k}q_i - \sum_{i=1}^{k}\theta_{-i}^{p}\sum_{j=1}^{i}q_j + \sum_{i=1}^{k+1}\theta_i^p\sum_{j=0}^{i-1}q_j + \sum_{i=k+2}^{k+p}\theta_i^p\sum_{j=0}^{k}q_j\right)}{\sqrt{\int J_c\left(s,r\right)^2 dr}\sqrt{\sum_{i=-k}^{k+p}\theta_i^2}}.$$

Proof of Proposition 3

As $u_t = \sum_{i=k}^{-k} q_{|i|}\epsilon_{t+i}$, then $\gamma_0 = E\left[u_t^2\right] = \sigma^2\sum_{i=-k}^{k}q_i^2$ and $\gamma_s = E\left[u_t u_{t-s}\right] = \sigma^2\sum_{j=-k+s}^{k}q_j q_{j-s}$ for $s = 1, 2, ..., 2k+1$. Therefore, in this case (24) and (25) becomes:

$$s_u^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 \to \gamma_0 = \sigma^2\sum_{i=-k}^{k}q_i^2$$

$$s_l^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 + 2T^{-1}\sum_{i=1}^{p}w\left(i,m\right)\sum_{t=i+1}^{T}\hat{u}_t\hat{u}_{t-i}$$

where

$$s_l^2 \to \gamma_0 + 2\sum_{i=1}^{p}w(i,m)\gamma_i = \sigma^2\sum_{i=-k}^{k}q_i^2 + 2\sigma^2\sum_{i=1}^{p}w\left(i,m\right)\sum_{j=-k+i}^{k}q_j q_{j-i}$$

hence substituting (24), (25), (50) and (51) into (22) and (23) the required result it is easily obtained.

Table 1. Scaling and shift terms for the ADF and PP asymptotic distributions in a seasonally adjusted random walk

		ADF Statistic				*PP* Statistics		
p	A	$\theta^p(1)$	$\sqrt{B}$	$\theta^p(1)/\sqrt{B}$	$A/\sqrt{B}$	m	Shift	Scale
0	0.174	1.000	0.909	1.100	0.191	0	0.174	0.826
1	0.043	0.929	0.907	1.024	0.048	1	0.115	0.885
2	-0.063	0.874	0.905	0.966	-0.070	2	0.096	0.967
3	-0.188	0.814	0.903	0.901	-0.209	3	0.033	0.904
4	0.223	0.998	0.879	1.135	0.254	4	-0.034	1.034
5	0.203	0.888	0.874	1.016	0.232	5	0.058	0.942
6	-0.076	0.835	0.873	0.957	-0.087	6	0.036	0.964
7	-0.176	0.788	0.871	0.904	-0.202	7	0.016	0.984
8	0.227	0.969	0.848	1.143	0.268	8	-0.005	1.005
9	0.012	0.847	0.841	1.007	0.015	9	0.027	0.973
10	0.033	0.858	0.842	1.019	0.039	10	0.020	0.980
11	-0.055	0.816	0.903	0.903	-0.061	11	0.012	0.988
12	0.220	0.936	0.818	1.143	0.269	12	0.006	0.994
16	0.178	0.892	0.798	1.118	0.223	16	0.011	0.989
20	0.161	0.867	0.782	1.108	0.206	20	0.009	0.991
40	0.100	0.789	0.738	1.070	0.135	40	0.004	0.996
100	-0.002	0.704	0.695	1.012	-0.002	100	0.002	0.998
200	0.002	0.681	0.680	1.002	0.003	200	0.001	0.999

Notes: The relevant terms for the ADF statistic are defined in (17), (20) and (21), for an augmentation lag of p, while the shift and scale terms for the PP statistic are given in (28) and (29), respectively, for bandwidth m. In both cases, the filter weights used for $q(L)$ are those of the quarterly symmetric linear X-11 filter.

Table 2. Empirical size and power of ADF test with MA innovations and OLS detrending

	c	$k2$	$k2\_f$	$k3$	$k3\_f$	$l4$	$l4\_f$	$l12$	$l12\_f$
$\theta = 0.5$	0	0.0465	0.0364	0.0511	0.0997	0.0554	0.0430	0.0474	0.0371
	1	0.0553	0.0412	0.0588	0.1216	0.0639	0.0460	0.0551	0.0406
	2	0.0546	0.0387	0.0638	0.1450	0.0756	0.0513	0.0588	0.0425
	5	0.0942	0.0663	0.1153	0.2549	0.1341	0.0901	0.1009	0.0708
	10	0.1965	0.1387	0.2899	0.5309	0.3311	0.2338	0.2187	0.1578
$\theta = -0.5$	0	0.0442	0.0325	0.0524	0.0539	0.0529	0.0729	0.0460	0.0309
	1	0.0516	0.0337	0.0582	0.0612	0.0609	0.0901	0.0534	0.0319
	2	0.0620	0.0364	0.0675	0.0713	0.0681	0.1017	0.0611	0.0343
	5	0.0969	0.0523	0.1201	0.1287	0.1277	0.1909	0.1024	0.0498
	10	0.1866	0.1049	0.2685	0.2822	0.2962	0.4066	0.2095	0.1069
$\Theta = 0.5$	0	0.0491	0.0305	0.2132	0.2621	0.2051	0.1973	0.0553	0.0330
	1	0.0559	0.0301	0.2711	0.3276	0.2620	0.2492	0.0642	0.0337
	2	0.0616	0.0318	0.3204	0.3877	0.3070	0.2924	0.0711	0.0358
	5	0.1066	0.0514	0.5105	0.5956	0.4956	0.4741	0.1272	0.0584
	10	0.2342	0.1152	0.8277	0.8866	0.8198	0.8016	0.2837	0.1421

Table 3. Empirical size and power of ADF test with MA innovations and GLS detrending

	c	$k2$	$k2\_f$	$k3$	$k3\_f$	$l4$	$l4\_f$	$l12$	$l12\_f$
$\theta = 0.5$	0	0.0502	0.0366	0.0545	0.1034	0.0619	0.0442	0.0503	0.0371
	1	0.0708	0.0517	0.0783	0.1503	0.0876	0.0628	0.0712	0.0532
	2	0.0958	0.0695	0.1119	0.2023	0.1215	0.0913	0.1002	0.0740
	5	0.1734	0.1311	0.2125	0.3531	0.2307	0.1788	0.1819	0.1410
	10	0.3068	0.2500	0.4191	0.5852	0.4572	0.3849	0.3349	0.2744
$\theta = -0.5$	0	0.0486	0.0303	0.0513	0.0538	0.0544	0.0762	0.0488	0.0280
	1	0.0728	0.0457	0.0809	0.0844	0.0844	0.1162	0.0736	0.0424
	2	0.0985	0.0610	0.1118	0.1163	0.1162	0.1583	0.1009	0.0585
	5	0.1749	0.1133	0.2076	0.2139	0.2187	0.2838	0.1812	0.1112
	10	0.3060	0.2173	0.4147	0.4243	0.4431	0.5221	0.3365	0.2252
$\Theta = 0.5$	0	0.0570	0.0310	0.2083	0.2439	0.2057	0.1994	0.0629	0.0345
	1	0.0798	0.0449	0.2899	0.3351	0.2867	0.2766	0.0896	0.0501
	2	0.1001	0.0566	0.3597	0.4114	0.3560	0.3433	0.1117	0.0632
	5	0.1801	0.1062	0.5450	0.6066	0.5444	0.5308	0.2043	0.1221
	10	0.3274	0.2182	0.7473	0.7915	0.7507	0.7397	0.3750	0.2553

Table 4. Empirical size and power of ADF test with MA innovations and Mean Recursive detrending

	c	$k2$	$k2\_f$	$k3$	$k3\_f$	$l4$	$l4\_f$	$l12$	$l12\_f$
$\theta = 0.5$	0	0.0778	0.0543	0.0795	0.1606	0.0865	0.0600	0.0785	0.0558
	1	0.0992	0.0711	0.1080	0.2154	0.1176	0.0835	0.1010	0.0736
	2	0.1300	0.0939	0.1398	0.2741	0.1523	0.1082	0.1328	0.0976
	5	0.2140	0.1575	0.2483	0.4309	0.2715	0.2020	0.2199	0.1638
	10	0.3965	0.3135	0.5092	0.7323	0.5576	0.4487	0.4248	0.3376
$\theta = -0.5$	0	0.0773	0.0441	0.0812	0.0846	0.0834	0.1183	0.0746	0.0408
	1	0.1071	0.0629	0.1080	0.1144	0.1131	0.1629	0.1097	0.0588
	2	0.1290	0.0747	0.1387	0.1456	0.1404	0.1997	0.1284	0.0680
	5	0.2165	0.1358	0.2370	0.2485	0.2464	0.3331	0.2218	0.1269
	10	0.3890	0.2626	0.4840	0.4971	0.5089	0.6203	0.4118	0.2598
$\Theta = 0.5$	0	0.0861	0.0425	0.3243	0.3796	0.3151	0.3043	0.0962	0.0483
	1	0.1145	0.0608	0.4138	0.4781	0.4038	0.3894	0.1307	0.0667
	2	0.1414	0.0746	0.4817	0.5507	0.4711	0.4557	0.1576	0.0834
	5	0.2373	0.1327	0.6907	0.7560	0.6828	0.6651	0.2674	0.1491
	10	0.4522	0.2824	0.9291	0.9560	0.9257	0.9170	0.5112	0.3265

Table 5. Empirical size and power of ADF test with MA innovations and WSLS detrending

	c	$k2$	$k2\_f$	$k3$	$k3\_f$	$l4$	$l4\_f$	$l12$	$l12\_f$
$\theta = 0.5$	0	0.0341	0.0208	0.0463	0.1049	0.0550	0.0353	0.0350	0.0232
	1	0.0466	0.0304	0.0626	0.1476	0.0758	0.0497	0.0503	0.0321
	2	0.0642	0.0428	0.0872	0.1928	0.1010	0.0675	0.0687	0.0465
	5	0.1182	0.0796	0.1762	0.3495	0.2044	0.1414	0.1327	0.0915
	10	0.2681	0.1942	0.4271	0.6757	0.4841	0.3716	0.3042	0.2231
$\theta = -0.5$	0	0.0317	0.0147	0.0429	0.0453	0.0472	0.0711	0.0331	0.0151
	1	0.0487	0.0237	0.0622	0.0666	0.0690	0.1042	0.0530	0.0249
	2	0.0623	0.0315	0.0818	0.0867	0.0886	0.1354	0.0648	0.0307
	5	0.1150	0.0622	0.1622	0.1712	0.1760	0.2524	0.1286	0.0608
	10	0.2562	0.1525	0.3968	0.4111	0.4339	0.5492	0.2910	0.1584
$\Theta = 0.5$	0	0.0409	0.0172	0.2572	0.3080	0.2525	0.2407	0.0497	0.0212
	1	0.0588	0.0264	0.3414	0.4084	0.3358	0.3210	0.0713	0.0334
	2	0.0731	0.0336	0.4124	0.4851	0.4059	0.3889	0.0894	0.0402
	5	0.1429	0.0678	0.6434	0.7197	0.6404	0.6216	0.1762	0.0856
	10	0.3298	0.1793	0.9229	0.9540	0.9247	0.9158	0.4057	0.2263

Table 6. Empirical size and power of VRT and PP with MA innovations and OLS detrending

	c	VRT	VRT_f	PPB	PPB_f	$PPQS$	$PPQS\_f$
$\theta = 0.5$	0	0.0671	0.0650	0.0983	0.0781	0.1415	0.1129
	1	0.0885	0.0845	0.1131	0.0879	0.1724	0.1345
	2	0.1064	0.1028	0.1305	0.1025	0.2051	0.1591
	5	0.1851	0.1768	0.2142	0.1754	0.3609	0.2838
	10	0.3391	0.3270	0.4256	0.3942	0.7010	0.6000
$\theta = -0.5$	0	0.0524	0.0522	0.0395	0.0373	0.0438	0.0439
	1	0.0673	0.0670	0.0396	0.0378	0.0463	0.0467
	2	0.0852	0.0849	0.0426	0.0397	0.0514	0.0514
	5	0.1510	0.1505	0.0794	0.0734	0.0955	0.0962
	10	0.2766	0.2748	0.2086	0.1942	0.2489	0.2511
$\Theta = 0.5$	0	0.0979	0.0976	0.5687	0.5586	0.5802	0.5680
	1	0.1273	0.1266	0.6841	0.6718	0.6959	0.6827
	2	0.1555	0.1544	0.7638	0.7537	0.7742	0.7627
	5	0.2590	0.2579	0.9311	0.9245	0.9371	0.9303
	10	0.4662	0.4641	0.9980	0.9975	0.9984	0.9979

Table 7. Empirical size and power of MSB, MZ_α and MZ_t with MA innovations and OLS detrending

		Panel A: MSB statistic			
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.4740	0.3528	0.3176	0.2392
	1	0.5964	0.4600	0.4177	0.3226
	2	0.6893	0.5418	0.4980	0.3944
	5	0.8945	0.7838	0.7462	0.6349
	10	0.9951	0.9775	0.9694	0.9254
$\theta = -0.5$	0	0.0326	0.0303	0.0368	0.0375
	1	0.0474	0.0442	0.0558	0.0560
	2	0.0615	0.0575	0.0692	0.0702
	5	0.1379	0.1289	0.1543	0.1554
	10	0.3590	0.3397	0.3970	0.3987
$\Theta = 0.5$	0	0.6534	0.6430	0.6535	0.6417
	1	0.7823	0.7712	0.7819	0.7691
	2	0.8570	0.8479	0.8553	0.8462
	5	0.9729	0.9696	0.9728	0.9688
	10	0.9997	0.9996	0.9997	0.9996

		Panel B: MZ_α statistic			
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.3959	0.2116	0.1748	0.0830
	1	0.5103	0.2890	0.2401	0.1096
	2	0.5912	0.3507	0.2928	0.1402
	5	0.8206	0.5754	0.5075	0.2754
	10	0.9816	0.8757	0.8408	0.6044
$\theta = -0.5$	0	0.2400	0.2269	0.2614	0.2637
	1	0.3131	0.2973	0.3420	0.3444
	2	0.3710	0.3503	0.4051	0.4079
	5	0.5709	0.5477	0.6142	0.6166
	10	0.8592	0.8377	0.8934	0.8945
$\Theta = 0.5$	0	0.7344	0.7151	0.7337	0.7126
	1	0.8453	0.8272	0.8428	0.8227
	2	0.9060	0.8919	0.9040	0.8881
	5	0.9846	0.9799	0.9838	0.9789
	10	0.9997	0.9996	0.9998	0.9996

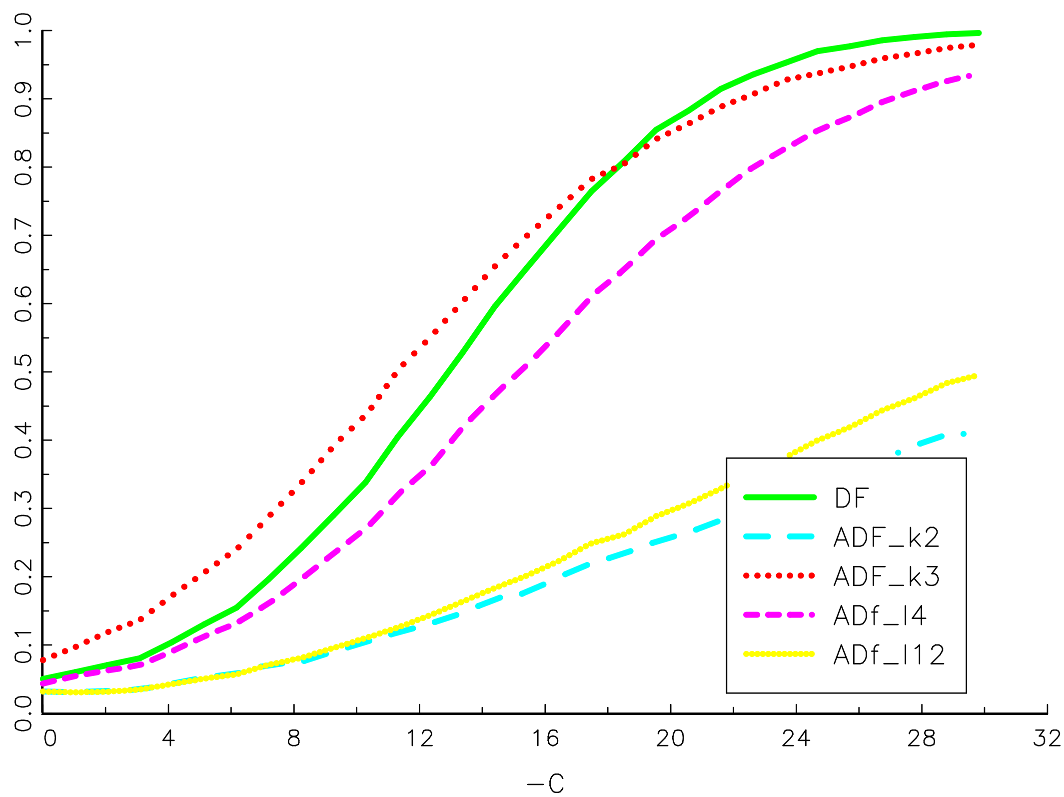
		Panel C: MZ_t statistic			
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.3252	0.1246	0.0899	0.0196
	1	0.4235	0.1667	0.1203	0.0298
	2	0.4979	0.2048	0.1547	0.0367
	5	0.7247	0.3768	0.2979	0.0850
	10	0.9494	0.6866	0.6233	0.2518
$\theta = -0.5$	0	0.5477	0.5308	0.5747	0.5775
	1	0.6087	0.5891	0.6397	0.6425
	2	0.6522	0.6326	0.6870	0.6883
	5	0.8130	0.7920	0.8456	0.8469
	10	0.9544	0.9415	0.9699	0.9701
$\Theta = 0.5$	0	0.7575	0.7306	0.7545	0.7248
	1	0.8509	0.8263	0.8460	0.8185
	2	0.9042	0.8846	0.9027	0.8799
	5	0.9806	0.9732	0.9795	0.9703
	10	0.9994	0.9989	0.9992	0.9985

Table 8. Empirical size and power of MSB; MZ_α and MZ_t with MA innovations and GLS detrending

		Panel A: MSB statistic			
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.3091	0.2307	0.2069	0.1614
	1	0.4169	0.3192	0.2854	0.2258
	2	0.5159	0.4033	0.3686	0.2955
	5	0.7295	0.6133	0.5751	0.4866
	10	0.8844	0.8181	0.7939	0.7302
$\theta = -0.5$	0	0.0365	0.0346	0.0408	0.0408
	1	0.0559	0.0536	0.0616	0.0617
	2	0.0741	0.0706	0.0809	0.0819
	5	0.1535	0.1478	0.1690	0.1699
	10	0.3588	0.3468	0.3868	0.3898
$\Theta = 0.5$	0	0.4881	0.4783	0.4881	0.4791
	1	0.6345	0.6243	0.6338	0.6234
	2	0.7191	0.7089	0.7192	0.7086
	5	0.8706	0.8644	0.8704	0.8634
	10	0.9565	0.9533	0.9564	0.9533
Panel B: MZ_α statistic					
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.2517	0.1437	0.1183	0.0608
	1	0.3490	0.2019	0.1661	0.0885
	2	0.4407	0.2659	0.2207	0.1232
	5	0.6500	0.4495	0.3896	0.2367
	10	0.8392	0.6908	0.6419	0.4756
$\theta = -0.5$	0	0.1658	0.1579	0.1794	0.1818
	1	0.2437	0.2341	0.2621	0.2644
	2	0.3216	0.3088	0.3458	0.3478
	5	0.5139	0.4961	0.5482	0.5527
	10	0.7471	0.7307	0.7805	0.7825
$\Theta = 0.5$	0	0.5687	0.5525	0.5678	0.5504
	1	0.7265	0.7108	0.7253	0.7077
	2	0.8104	0.7955	0.8096	0.7928
	5	0.9291	0.9206	0.9280	0.9188
	10	0.9813	0.9777	0.9809	0.9769

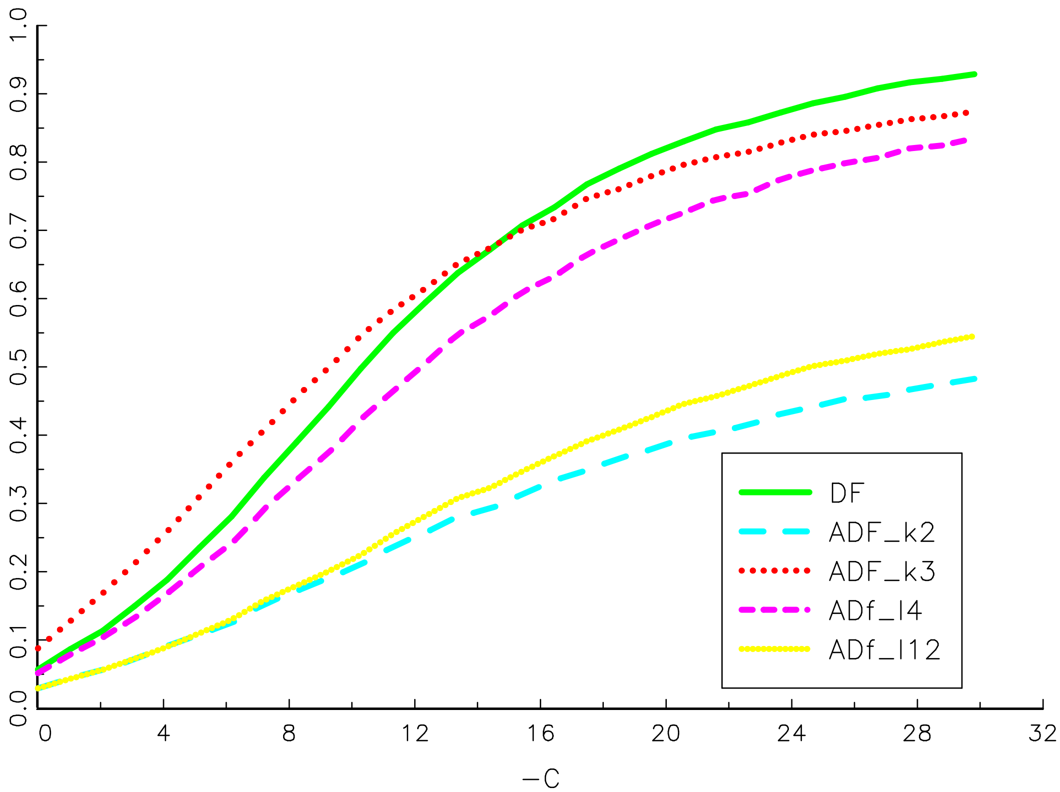
Panel C: MZ_t statistic					
	c	B	B_f	QS	QS_f
$\theta = 0.5$	0	0.2155	0.0959	0.0723	0.0238
	1	0.3011	0.1381	0.1037	0.0354
	2	0.3833	0.1831	0.1409	0.0497
	5	0.5855	0.3311	0.2659	0.1062
	10	0.7979	0.5714	0.5082	0.2689
$\theta = -0.5$	0	0.3024	0.2924	0.3234	0.3243
	1	0.4348	0.4217	0.4611	0.4621
	2	0.5435	0.5259	0.5737	0.5771
	5	0.7333	0.7166	0.7659	0.7682
	10	0.8841	0.8718	0.9083	0.9095
$\Theta = 0.5$	0	0.6097	0.5912	0.6101	0.5886
	1	0.7734	0.7521	0.7745	0.7496
	2	0.8511	0.8332	0.8516	0.8312
	5	0.9499	0.9412	0.9506	0.9396
	10	0.9884	0.9844	0.9880	0.9842

Figure 1: Seasonal adjustment and the asymptotic power function of the ADF test: OLS detrending



Notes: The DGP is (9) for $T = 400$. The DF test applied to the unadjusted data is denoted DF; otherwise results are for the ADF tests applied to quarterly data adjusted using the symmetric X-11 filter with k2, k3, l4 and l12 denoting automatic lag selection procedures described in Section 4. The nominal test size is 0.05.

Figure 2: Seasonal adjustment and the asymptotic power function of the ADF test: GLS detrending



Notes: As for Figure 1.

Figure 3: Seasonal adjustment and the asymptotic power function of the ADF test: Mean Recursive detrending

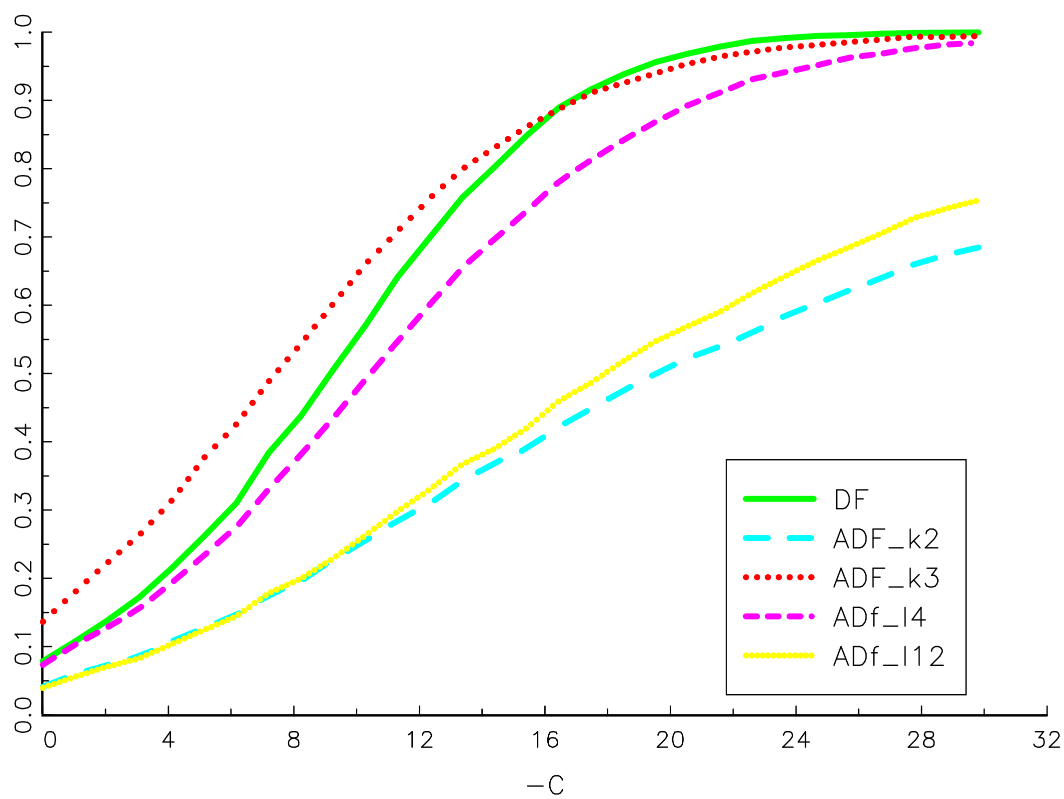


Figure 4: Seasonal adjustment and the asymptotic power function of the ADF test: WSLS detrending

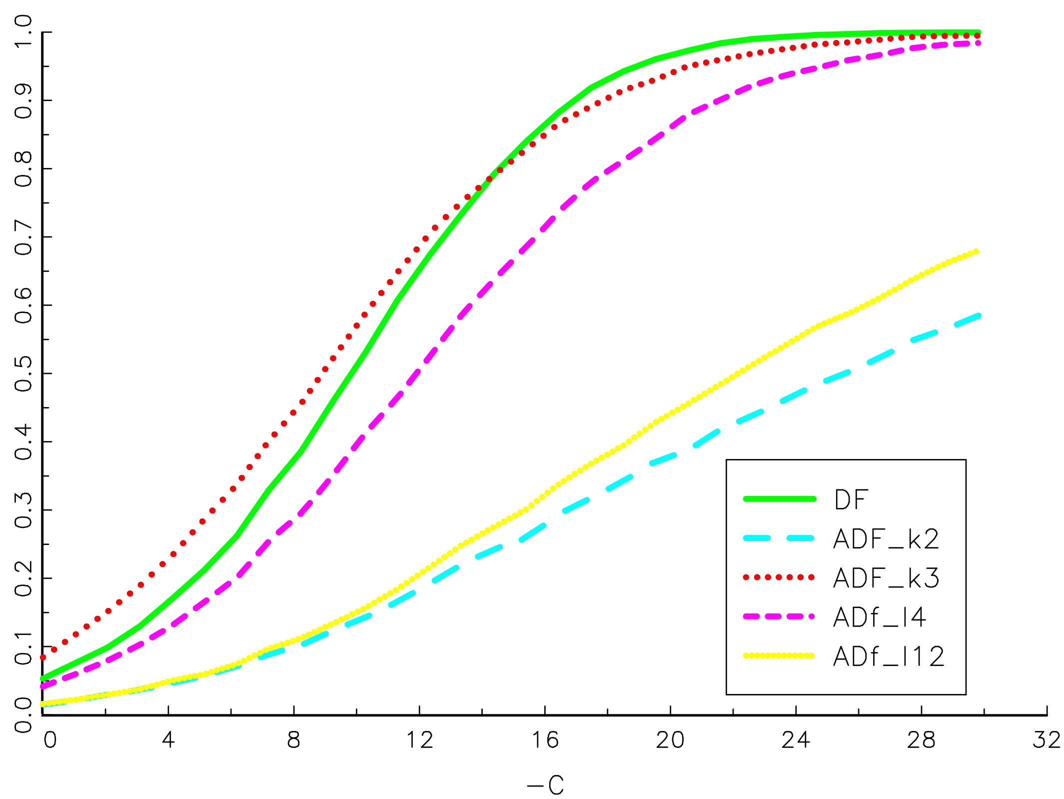


Figure 5: Seasonal adjustment and the asymptotic power function of the PP and VRT tests: OLS detrending

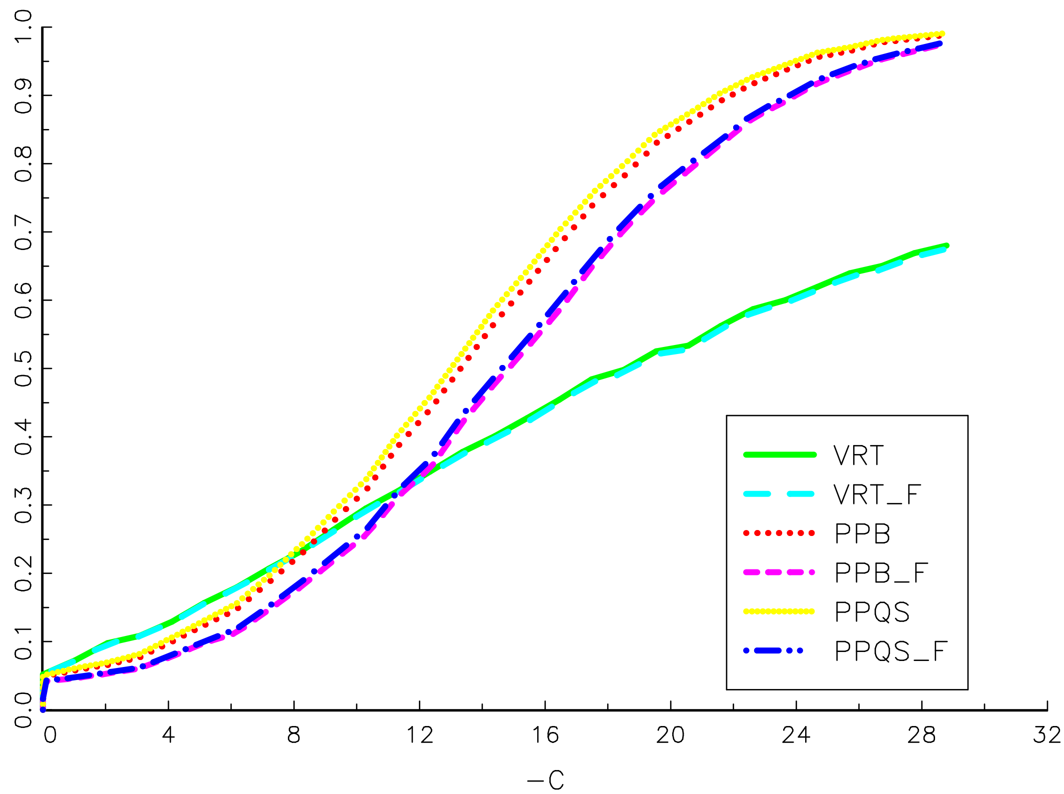


Figure 6: Seasonal adjustment and the asymptotic power function of the MSB test: OLS detrending

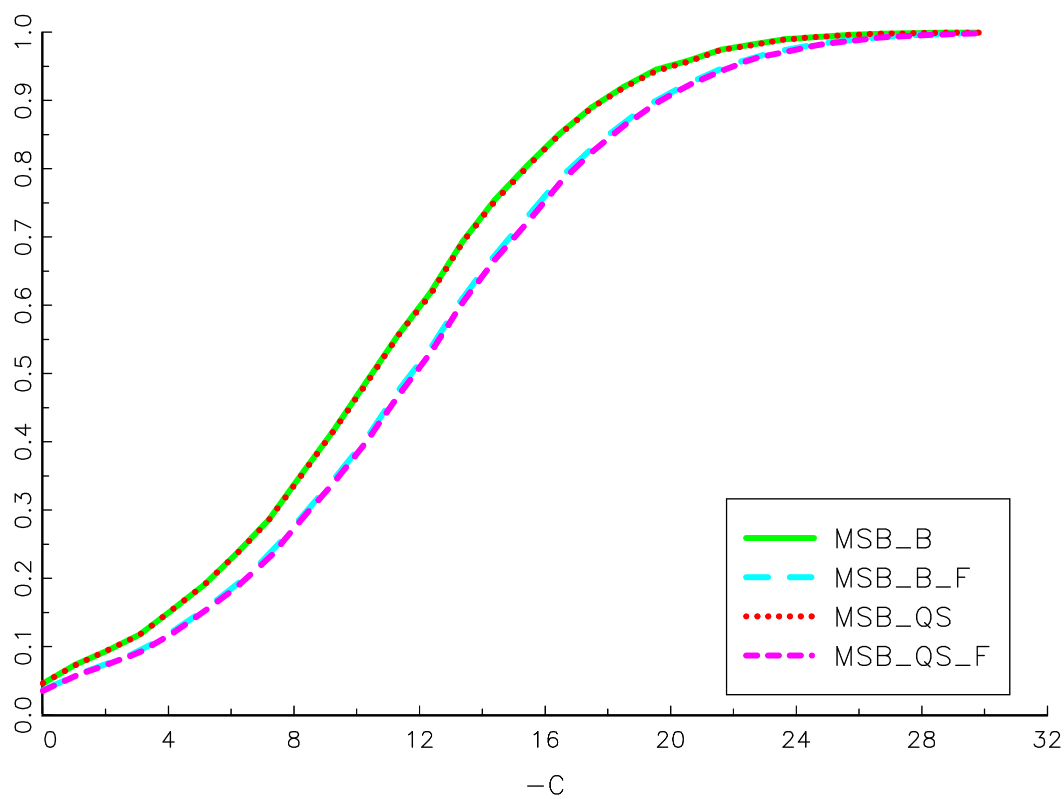


Figure 7: Seasonal adjustment and the asymptotic power function of the MSB test: GLS detrending

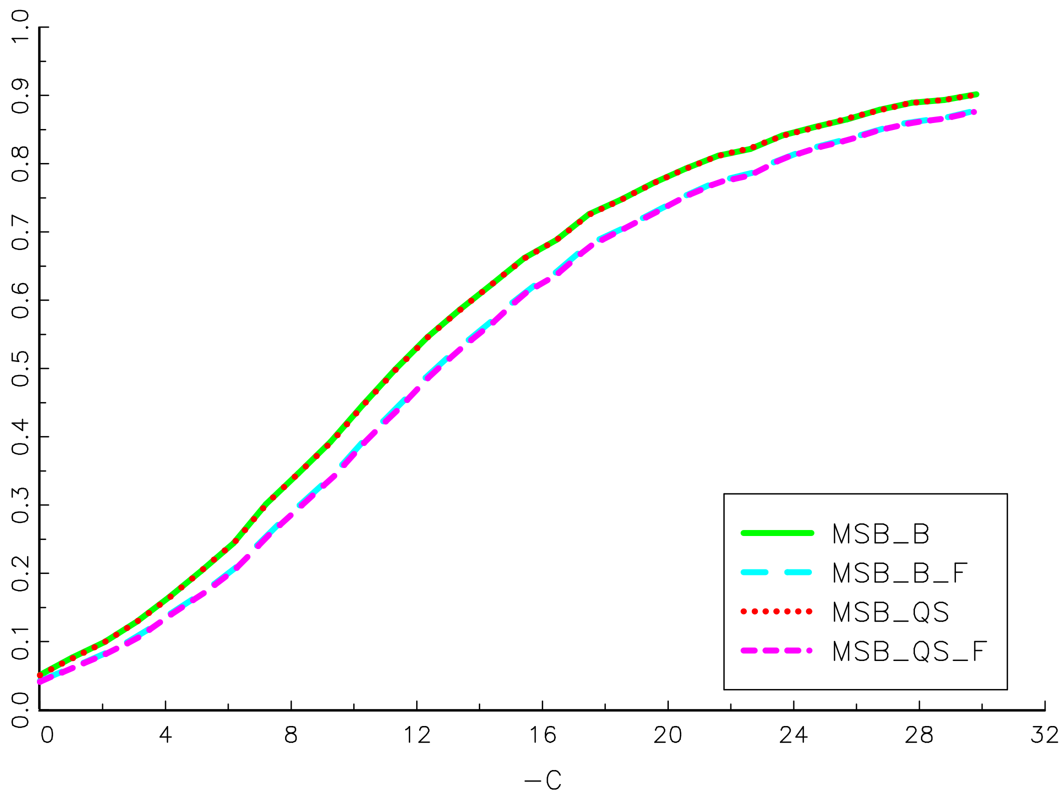


Figure 8: Seasonal adjustment and the asymptotic power function of the MZa and MZt tests: OLS detrending

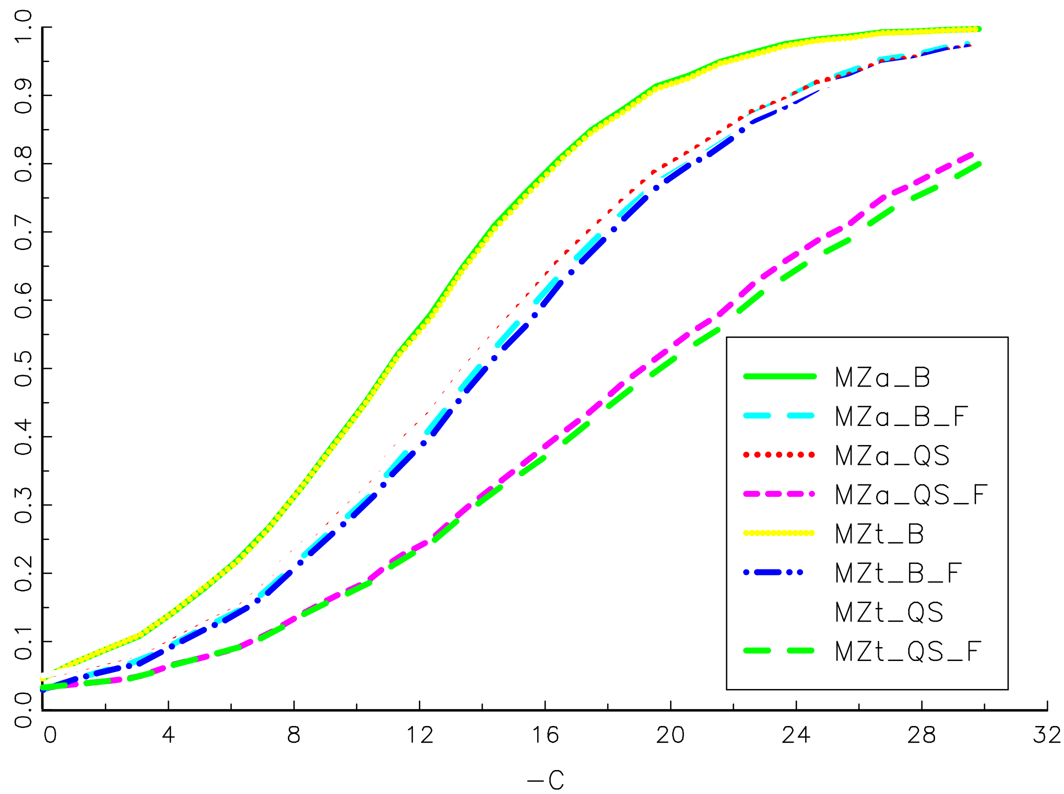


Figure 9: Seasonal adjustment and the asymptotic power function of the MZa and MZt tests: GLS detrending