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Characterizing monetary and fiscal policy rules and interactions when commodity prices matter

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Characterizing monetary and fiscal policy rules and interactions when commodity prices matter*

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Abstract

Using conventional rules to characterize policy behaviour in emerging market economies requires innovations capable of capturing distinctive structural characteristics. We examine the extent to which commodity price fluctuations matter for monetary and fiscal policy formulation in high primary commodity export economies. Markov mixture specifications of monetary and fiscal policy rules stylized to account for commodity price slacks are estimated using specifically designed Bayesian techniques. We find that policy authorities indeed respond to commodity price slacks but with variations depending on the policy regime in place and country. The results hold implications for the correct specification of policy rules and interactions in DSGE models for such economies.

Keywords: Monetary and fiscal policy rules, policy interaction, Markov-switching, unconstrained permutation Bayesian estimation, primary commodities.

JEL Classification: C51; E52; F41 E62

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1 Introduction

The ability of monetary and fiscal policy rules to adequately characterize the policy formation process should depend to a great extent on how well they factor in the distinctive structural characteristics of the economy they are designed to represent. What we have seen from the literature, however, is the persistent application of the more familiar workhorses for monetary and fiscal policy reaction functions. Monetary policy characterization delivered via Taylor (1993) type rules in which the behaviour of the central bank is described as a systematic reaction to inflation and the output gap, and fiscal policy via Leeper (1991)’s characterization of fiscal behaviour in terms of budget surpluses or deficits as a response to past government debt.

In this paper we examine the proposition that monetary and fiscal authorities react to commodity price fluctuations in the formulation of policy in economies with high shares of primary commodities in total export. In doing so we historically characterize policy behaviour into episodes of ‘active’ and ‘passive’ policy regimes and match the equilibrium outcomes of policy interactions for selected countries chosen on the basis of the share of primary commodities in merchandise exports. As far as we are aware this is the first study to offer a structured analysis of the connection between commodity price slacks and policy rule characterization for economies with distinctive structural characteristics. We offer new evidence for the proper specification of policy reaction functions in dynamic stochastic general equilibrium (DSGE) models suitable for high primary commodity export economies.

In recent times, there has been a renewed debate on whether policy should respond to other variables such as the exchange rate, asset prices and other indicators of the business cycle, (see for examples Taylor & Williams, 2010; Sargent, 2014; Siklos & Bohil, 2009; Bernanke & Gertler, 2001; Reinhart & Rogoff, 2009; Lubik & Schorfheide, 2007). In spite of the renewed interest, it seems that the literature has generally overlooked a potentially important component of the policy response menu, which could potentially enhance the macroeconomic stabilization roles of monetary and fiscal policies (including capital controls), especially in emerging market economies with significant shares of primary commodities in total export.\footnote{1At best, some selected studies have recognized the importance of commodity prices in the estimation of monetary policy rules only as a subsidiary variable. For examples, in Bernanke and Gertler (1999, 2001) and Clarida, Galí, and Gertler (1998), commodity prices are used as a part of the instrument vector in the GMM estimation of constant-regime monetary policy rules.}

Contemporary evidence indicates that commodity prices have certainly been associated with international macroeconomic volatility. For examples; Caballero, Farhi, and Gourinchas (2008), Belke, Bordon, and Hendricks (2014), and Hegerty (2016) show that the persistent global imbalances observed in recent decades, the sub-prime crises, the asset price bubbles and the debt crises that followed are all interconnected with commodity prices. The connection to commodity prices is not so surprising especially when one considers the global pattern of dependence on primary commodities depicted in Table 1. We can see that the average share of primary commodities in world merchandise exports in 1975 was 65.1 percent, and still as high as 48.2 percent in 2014, representing a moderate fall of about 25.9 percent over the last five
decades, with the Middle East/North Africa and sub-Saharan Africa having the highest shares.

Table 1: Indicators of dependence on primary commodities (% in merchandise export).

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<tbody>
<tr>
<td>Sub-Saharan Africa</td>
<td>7.1</td>
<td>3.6</td>
<td>24.2</td>
<td>18.6</td>
<td>35.57</td>
<td>40.4</td>
<td>9.7</td>
<td>14.4</td>
<td>76.7</td>
<td>77</td>
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<tr>
<td>North Africa &amp; Middle E</td>
<td>4.9</td>
<td>0.9</td>
<td>10</td>
<td>4.5</td>
<td>72.6</td>
<td>75.4</td>
<td>6</td>
<td>1.8</td>
<td>93.6</td>
<td>82.6</td>
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<tr>
<td>South Asia</td>
<td>7.6</td>
<td>1.6</td>
<td>33.2</td>
<td>13.3</td>
<td>1.3</td>
<td>16.5</td>
<td>6.9</td>
<td>2.7</td>
<td>49.1</td>
<td>34.1</td>
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<td>Latin America</td>
<td>5.2</td>
<td>1.8</td>
<td>37.2</td>
<td>19.3</td>
<td>23.1</td>
<td>11.6</td>
<td>11.9</td>
<td>8.7</td>
<td>77.4</td>
<td>41.4</td>
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<tr>
<td>Low-income countries</td>
<td>7.9</td>
<td>2.9</td>
<td>29.2</td>
<td>15.5</td>
<td>23.8</td>
<td>28.2</td>
<td>6.8</td>
<td>3.9</td>
<td>67.6</td>
<td>50.5</td>
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<tr>
<td>Middle-income countries</td>
<td>6.1</td>
<td>1.2</td>
<td>21</td>
<td>10.1</td>
<td>32.2</td>
<td>11.1</td>
<td>5.3</td>
<td>4.4</td>
<td>64.6</td>
<td>26.8</td>
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<tr>
<td>High-income countries</td>
<td>3.9</td>
<td>1.6</td>
<td>11.2</td>
<td>9.2</td>
<td>8.1</td>
<td>10.6</td>
<td>3.6</td>
<td>3.9</td>
<td>26.7</td>
<td>25.3</td>
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<tr>
<td>Average</td>
<td>3.9</td>
<td>1.6</td>
<td>11.2</td>
<td>9.2</td>
<td>8.1</td>
<td>10.6</td>
<td>3.6</td>
<td>3.9</td>
<td>65.1</td>
<td>48.2</td>
</tr>
</tbody>
</table>

Data and classification of countries by region follows the World Bank, World Development Indicators (2016).

Why might commodity prices be an important ingredient for monetary and fiscal policy formulation? Consider three main reasons. First is the so-called Prebisch-Singer hypothesis (Prebisch, 1950; Singer, 1950), which postulates that real commodity prices follow a downward circular trend. Second is the long cycles that characterize real commodity prices which create booms and busts in income and unemployment, and third is the tendency for volatility in real commodity prices and the fact that this volatility is often time-varying (see Harvey, Kellard, Madsen, & Wohar, 2010; Hadri, 2011). If these reasons are credible then they hold important implications for monetary and fiscal policy rule specification, budget sustainability rules, and the behaviour of business cycles in high primary commodity export economies.

These cyclical and volatile characteristics of commodity prices would naturally mean that their consequences in emerging market economies would be more pronounced than in an industrialized counterpart. This is mainly due to factors that should otherwise help to moderate the cyclical effects being undermined by structural, institutional and sometimes political considerations that economists’ often assume away. For examples, Frankel (2011); Kaminsky, Reinhart, and Végh (2005); Talvi and Végh (2005) show that capital flow imbalances, fiscal instability, monetary policy and the Dutch disease tends to be more pro-cyclical in less developed commodity producing countries than their industrialized counterparts.

This study concerns itself with the question; do monetary and fiscal authorities in high commodity export economies consider commodity price fluctuations in the process of policy formulation? There are economic reasons to expect that this should be the case. Take the scenario of a positive commodity price shock for example, this would often lead to a real currency

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2The empirical results from the literature on the tests of the Prebisch-Singer hypothesis are mixed. Depending on the specification used, some studies find evidence in support of the hypothesis, e.g., Grilli and Yang (1988), while others do not. For example, T.-H. Kim, Pfaffenzeller, Rayner, and Newbold (2003) use a difference-stationary specification and found that most commodity price series are nonstationary. Recent studies, e.g. Harvey et al. (2010), Belke et al. (2014) and Hadri (2011), use more robust specifications, which are capable of dealing with the time series properties of the series, and this is crucial for characterizing the behaviour of shocks and the appropriate policy response. Transitory (or stationary) series require different policy responses from non-transitory (non-stationary) commodity price series.
appreciation; taking the form of a nominal appreciation under a floating exchange rate regime or
the form of money inflows and hence inflation under a fixed exchange rate regime. On the fiscal
side, we might expect a resulting increase in government spending in response to the increased
income from the actual sales, taxes and royalties on the primary commodity.

If policy does not react in a countercyclical way, then typical symptoms such as; increases in
the price of non-traded goods relative to traded goods, high interest rates, reallocation of labour
away from the non-export commodity, and current account deficits are likely to be pervasive in
the domestic economy. These kinds of symptoms are often associated with a diagnosis of the
Dutch disease (see Sachs & Warner, 2001; Gelb, 1988, for further discussion). We conjecture
that because policy makers in high commodity export economies are aware of the structural
peculiarities they face, they follow monetary and fiscal policy formulation processes that take
into account international prices of the relevant commodities including them as instruments in
the policy reaction functions.

The purpose of this paper is twofold. First is to empirically test the validity of the proposition
that policy rules in high primary commodity export economies respond to commodity price
slack, and hence answer the question whether conventional, Taylor and Leeper type monetary
and fiscal policy rules have attained a ‘one-size-fits all’ status. Second is to characterize
equilibrium outcomes of the identified episodes of monetary and fiscal policy interactions, to
further understand if the policy rules are characterized by spells of regime-switches and also
make statements about the extent of coordination or synchronization between monetary and
fiscal policy rules. We consider a sample of five high primary commodity export emerging
economies including; Chile, Brazil, Mexico, Nigeria, and South Africa and also include Canada
as an industrialized counterpart economy to benchmark the results and check for possible
systematic differences in the observed patterns for emerging markets and developed economies.

To achieve the specified goals, we modify Taylor’s, and Leeper’s specifications of monetary
and fiscal policy reaction functions in such a way that allows us to incorporate the possibility
that high primary commodity export economies consider commodity price slacks in formulating
policies. Further, to accommodate possible endogenous stochastic changes in the characterization
of these policy rules, we use simultaneous Markov switching specifications in the parameters and
residuals of the model. Our estimation approach allows us to avoid the weaknesses of maximum
likelihood estimation of Markov switching regressions by employing Bayesian estimation methods,
using specifically designed Markov chain Monte Carlo (MCMC) based sampling schemes. Finally,
equilibrium policy interaction outcomes are then evaluated using a framework following Davig

It is worth mentioning a caveat on the interpretation of the results from our design. Because
our analysis focuses on characterizing historical policy reaction functions, they are ‘simple’ or
adhoc rules and do not necessarily provide evidence on the optimality of the rules. Though
recent research as summarized in Taylor and Williams (2010), has shown that in most cases
simple rules are robust and actually work better in the real world, the rules estimated here hold
more potential for practical applications.\footnote{One of the main advantages of focusing on simple rules, from a policy perspective, is that it improves communication and commitment to the rule by the policy authority, mainly because the public can easily monitor the extent of implementation of the rules.}

Overall, we find that policy authorities in all the countries under investigation respond to commodity price slacks in different ways and depending on the policy regime in place. We also find that policy behaviour in these countries are characterized by distinctive episodes of ‘active’ and ‘passive’ policy regimes, driven by the response of interest rates to inflation, and tax revenues to past government debt. In particular, the monetary authorities in Chile, Canada, Mexico and South Africa respond to commodity price slacks in a counter-cyclical manner by tightening monetary policy, whereas the monetary response in Brazil and Nigeria seems to be rather pro-cyclical. On the fiscal side the evidence indicates that only Canada responds to commodity price booms in a counter-cyclical manner, whereas the fiscal response in Chile, Nigeria, Brazil and South Africa is rather pro-cyclical. Further to this we find that the implied inflation target when compared with the actual target announced by the respective central banks indicates that the central bank’s reaction to inflation is often not “hawkish” (aggressive) enough to achieve the announced target. Lastly, there is no evidence of policy synchronization between monetary and fiscal policy regime changes in all the sampled economies.

The robust and consistent evidence on the relevance of commodity price slacks and the stochastic regime changing nature of the policy rules estimated, holds at least two interesting implications for the correct specification of macroeconomic models considered suitable for policy analysis in high resource export economies. First, has to do with the implied idea that the Taylor rule; in which monetary policy reacts to inflation, the output gap, and the exchange rate (for small open economies), has attained a one-size-fits all status; often used as the theoretical default in the literature. Our results suggest that for emerging market economies, with structural characteristics leaning towards reliance on primary commodity export, the inclusion of a commodity price slack component, provides a better characterization of policy. Secondly, the regime switching behaviour of the policy rules reveals the shortcomings in earlier studies of monetary-fiscal policy interactions within New-Keynesian models in which constant regime specifications were used for both monetary and fiscal policy rules. Our study offers empirical support for the recent contributions in the literature with regime-switching specifications of monetary and fiscal policy rules in agent optimization based New-Keynesian models, see for examples Liu, Waggoner, and Zha (2011); Chung, Davig, and Leeper (2007); Davig and Leeper (2011).

The rest of this paper is laid out as follows. Section 2 highlights the connection of our paper with the literature on policy reaction functions and their interactions. In Section 3 we outline the specification of the monetary and fiscal policy rules used and discuss issues of identification. Section 4 contains the empirical strategy and the procedure for the implementation of the Bayesian simulation based techniques. In Section 5 we present and discuss the results from the Bayesian estimation procedure. Section 6 concludes.
2 Relevant literature

The seminal papers by Bernanke and Gertler (1999, 2001), and Clarida et al. (1998) were some of the initial studies to consider the role of primary commodities in monetary policy reaction functions, albeit as auxiliary variables. Specifically, Bernanke and Gertler (1999) address the question of whether and how central banks should respond to asset prices within a simulation and estimation based framework, they extend the small-scale New-Keynesian model with financial accelerator effects in Bernanke, Gertler, and Gilchrist (1999) by allowing for exogenous bubbles in asset prices. Their results indicate that a monetary policy regime that focuses on asset prices rather than economic fundamentals might be actively destabilizing because the monetary authority would be following the wrong indicator in such a situation. They empirically validate their theoretical model by applying the methods of Clarida et al. (1998) to estimate a forward-looking reaction function for the Federal Reserve Bank and the Bank of Japan using GMM techniques. Lagged values of the Dow-Jones commodity price index, among other variables are introduced as instruments in the estimation process. The results suggest that the Fed does not react to changes in stock prices, whereas the Bank of Japan appeared to have reacted to stock prices after the stock market crash of 1990.

The literature on monetary and fiscal policy reaction functions and interactions for emerging economies is sparse when compared to those that focus on industrialized economies. Nevertheless, some recent studies in Asia and Latin America contain relevant findings. For example, Corbo, Landerretche, and Schmidt-Hebbel (2002) find that central banks in Latin American countries tend to consider other non conventional objectives beyond inflation stabilization in setting interest rates. Mohanty and Klau (2005) use a standard Taylor type open economy reaction function estimated by OLS and GMM techniques to test whether central banks in emerging market economies react to changes in inflation, the output gap and the exchange rate, in a consistent and predictable manner. Their results show that in most emerging market economies, the interest rate responds strongly to the exchange rate and that asymmetries also exist in the degree of response to both positive and negative inflation shocks.

Empirical studies of policy reaction functions are often confronted with issues of non-linearities, structural breaks and time-varying behaviour. For example, Muscatelli, Tirelli, and Trecoci (2002) find evidence of multiple breaks in the estimated policy rules for many countries over a thirty year period. To account for these kinds of problems, Assenmacher-Wesche (2006) estimate policy rules for selected industrialized countries, using Markov-switching models that allow for shifts in the coefficients as well as residual variance. Similarly, Sims (1999) estimate a three state Markov-switching model with simultaneous switching in the coefficients and variances.

4 They explain that it is dangerous for a central bank to purse a policy that simultaneously responds to stock prices and inflation. If policy responds aggressively to inflation, i.e., it follows the Taylor principle, then it is not necessary for policy to also respond to asset prices.

5 In response to these set of papers, Cecchetti, Genberg, Lipsky, and Wadhwani (2000) conduct exploratory experiments on the Bernanke and Gertler model and show that the model is not robust to many modifications including; introducing interest rate smoothing, varying leverage levels, output gap, among others. They provide descriptive evidence to show that monetary policy should and has in fact reacted to asset prices in the U.S.
to accommodate stochastic endogenous regime changes.

Another relevant strand of the literature on policy rules is the notion that government debt could serve as an anchor for inflation and economic stability, termed the “fiscal theory of the price level (FTPL)”, popularized in a series of papers by Woodford (1995, 2001). This literature argues that a potentially stabilizing monetary-fiscal policy regime is one that combines a Taylor rule for monetary policy with nominal-deficit targeting for fiscal policy. In a related paper, Leeper (1991) examines the equilibrium conditions under which the intertemporal budget constraint is satisfied when there is a shock to real government debt. The question is whether or not shocks to government debt can provide a nominal anchor for money? Using simple monetary and fiscal policy rules, the paper provides analytical conditions that are required for a unique saddle path that pins down the inflation and government debt processes. Similarly, Woodford (2001), and Davig and Leeper (2011) emphasize the importance of coordination between monetary and fiscal policy rules in such a way that stability in the dynamic paths of the control variables are guaranteed. In particular, Woodford shows that even when monetary and fiscal policy is consistent with stable prices, there may be situations where agents expectations would coordinate on a different path. In order to exclude this possibility, it is necessary to commit to a fiscal policy that is locally Ricardian, the case where the primary budget evolves exogenously until certain debt limits are reached.

In a series of papers, Chung et al. (2007); Davig and Leeper (2006, 2011) explore the implications of monetary and fiscal policy interactions in a more realistic environment where policy is characterized by spells of stochastically determined Markov regime changes. In contrast to the conventional single-regime analyses, with stochastic regime changes, an agent’s decision rule embeds the probability that policies will change in the future, and this leads to an outcome where monetary and fiscal (tax) shocks always produce wealth effects. Drawing from these results, recent empirical studies on simple rules for monetary-fiscal policy interactions have cast the problem in a regime switching context, estimating the rules by maximum likelihood techniques. For example, Cevik, Dibooglu, and Kutan (2014) study the interactions between monetary and fiscal polices in emerging European economies using Markov-switching specifications estimated by maximum likelihood techniques. Their results suggest that monetary and fiscal policy rules in most emerging European economies exhibit switching properties between active and passive policy regimes.

One common concern in the estimation of Markov-regime switching policy rules is the possibility that there may be very few transitions between regimes, and this creates problems for the accurate estimation of the parameters in the Markov switching model, especially when it is done by maximum likelihood methods. In addition, there is also the problem commonly referred to as the ‘curse of dimensionality’ arising from the explosion in the number of parameters to be

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6Leeper (1991) shows that the sufficient condition for the existence of a unique saddle path equilibrium for the nominal anchor is the satisfaction of the Blanchard and Kahn conditions. That is, one root of the system should lie inside the unit circle and the other root should lie outside the unit circle.

7Other examples along these lines include: Chuku (2012); Semmler and Zhang (2004); Owyang and Ramey (2004), and Hutchison, Sengupta, and Singh (2013)
estimated in a Markov regime switching model. We carefully avoid these maximum likelihood based shortcomings by approaching the estimation of the monetary and fiscal policy rules from a specifically designed Bayesian inference techniques (see Frühwirth-Schnatter, 2004; Kaufmann, 2000; C.-J. Kim & Nelson, 1999a). Lastly, this paper contributes to the literature by considering bespoke policy reaction functions which are more suitable for economies that have high shares of primary commodities in merchandise export.

3 Policy rule specification and identification

3.1 The monetary policy rule

To characterize monetary policy rules in high commodity export economies, we adapt the general class of monetary policy rules known as the Taylor rule (see Taylor, 1993). According to this rule, ‘good’ monetary policy requires the central bank to react in a systematic manner to deviations in the observed inflation rate from the set target and to deviations of output from its potential level. One aspect of the adaptation involves the inclusion of the ‘primary commodities slack’ into Taylor’s baseline specification, and then testing if such an introduction improves the model fit for the selected countries. Taylor’s baseline characterization of the monetary policy reaction function for the Federal Reserve is given as:

\[
i_t^T = \bar{r} + \pi^* + \alpha_\pi (\pi_t - \pi^*) + \alpha_y (y_t - y_t^*),
\]  

(1)

where \(i_t^T\) is the policy interest rate, \(\bar{r}\) is the equilibrium real rate, \(\pi_t\) and \(\pi^*\) are the inflation rate and its target value respectively, and \((y_t - y_t^*)\) is the output gap. By assigning hypothetical values of 0.5 each to \(\alpha_\pi\) and \(\alpha_y\), Taylor shows that this rule approximates U.S monetary policy in the past several years. This formulation has been used to examine central banks’ reaction functions across several countries (see Siklos, 2008, for a review). For monetary policy to be stabilizing, the central bank is expected to react according to the so called “Taylor principle” where the weight on inflation exceeds unity, i.e., \((\alpha_\pi > 1)\), and the weight on the output gap is positive, \(\alpha_y \in (0, 1)\). This principle implies that when the estimated coefficient on inflation is greater than unity, the monetary authority responds to above target inflation by increasing interest rates. Using Leeper (1991)’s terminology, it is dubbed “active monetary policy”.

We consider recent theoretical and empirical refinements to the baseline Taylor rule. First is the idea that the policy maker conducts monetary policy based on future expectations of inflation, building on this, Clarida et al. (1998) proposed the estimation of a forward-looking rational expectations version of the Taylor rule. Second is the observation that monetary policy rules augmented with exchange rate movements outperform the conventional ones especially in small open economies (see Lubik and Schorfheide (2007), and Svensson (2000) for empirical and theoretical evidence). The third factor we consider is that central banks often follow a gradual mechanism in the adjustment of interest rates in order to avoid abrupt disruptions in
the economy possibly caused by sudden large changes in interest rates. This is the so-called “interest rate smoothing” mechanism (see Woodford, 2002; Goodfriend, 1991). This process implies that interest rates will be autocorrelated over time.

Putting these refinements together, we augment the baseline Taylor specification by introducing commodity price slacks, changes in the nominal exchange rate, rational expectations behaviour, and autocorrelation arising from interest rate smoothing, so that the specification becomes:

\[ i_t = (1 - \rho) \left\{ \bar{r} + \pi^* + \alpha_\pi (E\pi_{t+k} - \pi^*) + \alpha_y (Ey_{t+p} - y^*_{t+p}) + \alpha_{cm} (Ec_{t+j} - c^*_t) + \alpha_{ex} \Delta \epsilon_t \right\} + \rho \mu_{t-i} + \epsilon_t, \]

where \( E \) is the expectations operator, \( \{c\} \) is a measure of the relevant commodity price sequence, and \( \Delta \epsilon_t \) is the first difference of exchange rate. To obtain the estimation equation we carry out two more transformations: as in Clarida et al. (1998), we eliminate the unobserved forecast errors of inflation, the output gap and the commodity price slack, and the error term \( \epsilon_t \) is defined as a linear combination of the forecast errors of inflation, the output gap and the commodity price slack, thus; \( \epsilon_t \equiv -(1 - \rho) [\alpha_\pi (\pi_{t+k} - E\pi_{t+k}) + \alpha_y (x_{t+p} - Ex_{t+p}) + \alpha_{cm} (c_{t+j} - Ec_{t+j})] + \nu_t \), where \( \nu_t \) is an exogenous disturbance.

To accommodate non-linearities and possible stochastic regime switching behaviour of monetary policy rules, we specify the Markov-switching version of the model thus:

\[ i_t = \left[ 1 - \rho(S^M_t) \right] \left\{ \alpha_0 (S^M_t) \pi_t + \alpha_x (S^M_t) x_t + \alpha_{cm} (S^M_t) c_t + \alpha_{ex} (S^M_t) \Delta \epsilon_t \right\} + \rho \mu_{t-i} (S^M_t) + \epsilon_t (S^M_t), \]

where \( (S^M_t) = 1, \ldots, K \) are the unobservable \( K \) states of the monetary policy regime, with transition probability \( P^M \). In the Markov switching specification, the limiting assumption imposed is that there are two policy regimes; ‘active’, and ‘passive’. There is no restriction on the sign, magnitude or timing of the switching coefficients in (4). Furthermore, as in

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8Other specific reasons why the central bank will rather not suddenly adjust interest rates to their desired target levels include: the fear of disruption to financial markets (Goodfriend, 1991), to maintain its ability to influence aggregate demand by means of small changes in the policy rate, to minimize the risks of policy mistakes especially in a data limited environment, to mitigate the effects of transaction frictions, and to avoid falling below the interest rate lower bound trap (see Woodford, 2002).

9As in Clarida et al. (1998), this specification is based on the assumption that in the constant-regime model, interest rates, inflation, and the commodity slack component are \( I(0) \) variables, whereas, this is not required to hold in the nonlinear specification. Preliminary Dickey-Fuller tests for unit root on the variables confirm that this holds for the relatively short samples considered for the selected countries.
Assenmacher-Wesche (2006) and Sims (1999), the error term is also allowed to switch between low and high variances simultaneously with the coefficients, so that:

\[ \epsilon_t \sim N(0, \sigma^2(S_t)) \]
\[ \sigma^2(S_t) \in [\sigma_1^2, \sigma_2^2]; \quad \sigma_1^2 < \sigma_2^2. \] (5)

The switching nature of the error term is used to capture idiosyncrasies in the policy implementation technology and unmodeled or non-economic shocks such as the financial crisis or changes in institutional dynamics. An additional advantage of the forward-looking rational expectations approach by Clarida et al. (1998), is that it enables us to back-out the implied inflation target during each monetary policy episode. By assuming that the long-run equilibrium real interest rate \( \bar{r}^* \) is independent of monetary policy and equal to the ex post average rate, it is possible to back out the implicit inflation target as\(^{10}\):

\[ \pi^*(S^M_t) = \bar{r}^* - \alpha_0(S^M_t) \alpha_\pi(S^M_t) - 1. \] (6)

### 3.2 The fiscal policy rule and the policy mix

Although there are no generally established fiscal policy rules in the literature, Leeper (1991), and Bohn (1998)’s specifications are about the most commonly cited. In Leeper’s baseline specification, fiscal policy is characterized by government adjusting taxes (or rates) in response to the past values of real debt. We augment Leeper’s rule by including the commodity price slack, output gap and government expenditure variables. Further, to account for gradual tax (revenue) smoothing behaviour we also allow for non-linearities in the form of possible regime switches. To facilitate this, we introduce an autoregressive term within a first-order Markov-switching specification so that the estimation equation is given as:

\[ \tau_t = \gamma_0(S^F_t) + \gamma_1(S^F_{t-1}) + \gamma_2(S^F_{t})x_t + \gamma_3(S^F_{t})g_t + \gamma_4(S^F_{t})c_t + \rho(S^F_{t})\tau_{t-1} + \epsilon(S^F_{t}). \] (7)

where \( \tau_t \) is a measure of the share of revenue (tax) in GDP, \( b_{t-1} \) is the lagged value of public debt in GDP, \( x_t \) is the output gap, \( g_t \) is government expenditure in GDP, and \( c_t \) is the commodity price slack.

In line with the terminology and characterization by Leeper (1991) and Davig and Leeper (2006), fiscal policy is couched ‘passive’ (or in Woodford (2001)’s terminology Ricardian), if taxes are increased enough in response to the debt-GDP ratio to offset interest payments, and satisfy the intertemporal budget constraint (IBC). For this condition to hold the estimated coefficient on the lagged debt-GDP ratio is expected to be positive,\(^{11}\) \( \gamma_1 > 0 \), so that an increase in the stock of public debt outstanding leads to an increase in tax revenues, and hence a decrease

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\(^{10}\)Note that we have defined \( \bar{r}^* = \bar{r} + \pi^* \). The theoretical derivations including justification for this application can be found in Valente (2003) and Favero and Rovelli (2003).

\(^{11}\)In a stricter sense, the conditions that characterize active and passive fiscal policy rules are; \( \gamma_1 \in (0, 2/\pi^*) \) and \( \gamma_1 \notin (0, 2/\pi^*) \) respectively (see Schmitt-Grohé & Uribe, 2007).
in the government budget deficit. Conversely, when $\gamma_1 \leq 0$, fiscal policy is said to be active and therefore not constrained by the current budgetary requirements of the fiscal authority. The unobserved state variables in the monetary and fiscal policy rules, $(S^M_t$ and $S^F_t$), both follow a two-state first-order Markov-switching process described as follows:

$$
P[S_t = 1|S_{t-1} = 1] = \xi_{ii}, \quad P[S_t = 1|S_{t-1} = 2] = 1 - \xi_{ii} \\
P[S_t = 2|S_{t-1} = 1] = 1 - \xi_{jj}, \quad P[S_t = 2|S_{t-1} = 2] = \xi_{jj},
$$

where $\xi$ are the ergodic transition probabilities of remaining in the same state or moving to a different state. In other words, it is a measure of the persistence of the states. Consequently, the expected duration of the states is a random variable following a geometric distribution with parameter $1 - \xi_{jj,ii}$ given by:

$$
\mathbb{E}(D) = \frac{1}{1 - \xi_{jj,ii}}.
$$

To study monetary-fiscal policy interactions, let $\hat{S} = (S^M_t, S^F_t)$ denote the joint monetary-fiscal policy states. Then, following Davig and Leeper (2006, 2011), the joint distribution of policy regimes evolves according to a Markov chain with transition matrix $\hat{P} = P^M \otimes P^F$, where $P^M$ and $P^F$ are the transition probability matrices for monetary and fiscal policy regimes respectively. Given the first-order, two-state regime specification for monetary and fiscal policy rules, the joint policy process can be classified into four possible equilibrium outcomes. In Table 2 the equilibrium outcomes for all possible combinations of policy interactions are depicted. The probabilities on the main diagonal of the inner table represents the outcome for; active fiscal and active monetary policy (AF/AM) interaction, and passive fiscal and passive monetary policy (PF/PM) interactions respectively. When both monetary and fiscal policies are active, then their interaction does not guarantee a sustainable path for the price level and the budget balance. Under these conditions the policy mix is said to be explosive. Conversely, when they are both passive the policy mix is in the region of indeterminacy. The most common specification used in standard DSGE models is the ‘active monetary policy-passive fiscal policy’ (AM/PF) combination, in which case the policy mix is said to be Ricardian. Finally, when price level determination is based on conditions that are required to satisfy the government budget constraint, then we live in the world of the “fiscal theory of the price level” where fiscal policy is active and monetary policy is passive (AF/PM). Our objective is to identify the equilibrium outcomes of policy interaction, and provide a historical narrative of the joint monetary-fiscal interactions.

<table>
<thead>
<tr>
<th>Fiscal Policy</th>
<th>Monetary policy</th>
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<tbody>
<tr>
<td></td>
<td>Active</td>
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<td>Passive</td>
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<tr>
<td>Active</td>
<td>Explosive periods (AF/AM)</td>
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<tr>
<td>Passive</td>
<td>FTPL periods (AF/PM)</td>
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<tr>
<td></td>
<td>Ricardian periods (PF/AM)</td>
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<tr>
<td></td>
<td>Indeterminacy periods (PF/PM)</td>
</tr>
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</table>

Table 2: Equilibrium outcomes of monetary-fiscal policy interactions
policy episodes for the selected high primary commodity export economies.

4 Empirical strategy and implementation

In this section, the key aspects of practical Bayesian inference for the type of Markov-switching regressions we estimate are described. In particular we discuss issues concerning the choice of priors, estimation of the switching parameters, filtering and smoothing techniques for the transition probabilities, complete data likelihood estimation and sampling procedures including model diagnostics and specification testing.

4.1 Parameter estimation of the Markov mixture model

The general form of the Markov switching regression considered is as follows:\(^\text{12}\) let \(y^N = (y_1, \ldots, y_N)\) denote the \(N \times 1\) vector of the dependent variable, So that the estimation equation is

\[
y_t = X_t \beta(S_t) + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_{\epsilon_t}^2),
\]

where \(X_t\) is a vector of explanatory variables which may also contain lagged dependent variables. The latent variable \(S_t\) is the process governing the Markov chain distribution, for which we impose the most flexible assumptions. Particularly, it is a first-order inhomogeneous Markov chain, with the conditional distribution of \(S_t\) being dependent on the history \(y_{t-1}\) and on its most recent past value \(S_{t-1}\). Hence,

\[
Pr(S_t = k|S_{t-1}, y^{t-1}) = Pr(S_t = k|S_{t-1}, y_{t-1}); \quad \forall k \in \{1, \ldots, K\}.
\]

The stochastic properties of \(S_t\) are sufficiently described by the \((K \times K)\) transition matrix \(\xi\), where each element \(\xi_{jk}\) of \(\xi\) represents the transition probabilities of moving from state \(j\) (say active) to state \(k\) (say passive).\(^\text{13}\) That is, \(\xi_{jk} = Pr(S_t = k|S_{t-1} = j)\). To define the complete data likelihood function, let \(\vartheta = (\beta_S, \sigma_S, \xi)\) be the collection of all the model parameters and transition probabilities, then the density \(p(\mathbf{y}, \mathbf{S}|\vartheta)\) of the joint distribution of \(\mathbf{S} = \{S_t\}^T_0\) and \(\mathbf{Y} = \{Y_t\}^T_0\) is given as:

\[
p(\mathbf{y}, \mathbf{S}|\vartheta) = p(S_0|\vartheta) \prod_{t=1}^T p(y_t|y_{t-1}, S_t, \vartheta)p(S_t|S_{t-1}, \vartheta) \label{eq:complete}\]
where \( p(y_t | y^{t-1}, S^t, \vartheta) = \sqrt{\frac{1}{2\pi\sigma_{S_t}}} \exp \left\{ -\frac{(y_t - X_t \beta_{S_t})^2}{2\sigma_{S_t}} \right\} \) is the Gaussian one-step ahead predictive density of the conditional distribution of \( y_t \), knowing past realizations \( y^{t-1} \) and the states \( S^t \) and \( p(S_t | S^{t-1}, y^t, \vartheta) \) is the density of the conditional distribution of the states \( S_t \).

Maximum likelihood (ML) techniques are commonly used to estimate the parameters in (12); a popular choice being the expectations-maximization (EM) algorithm introduced by Dempster, Laird, and Rubin (1977).\(^{14}\) One of the problems with this method however, is that it is often difficult to find the global maximum of the likelihood numerically, especially when the sample size is small or when the different states are not clearly distinct (see Karlis & Xekalaki, 2003). Furthermore, as in any incomplete data problem, the provision of standard errors is not straightforward within ML estimation, this could be partly explained by the often (near) singularity of the matrix of second partial derivatives of the log likelihood. More concerning is the theoretically well established understanding that the sample size, \( N \), has to be very large before asymptotic theory of maximum likelihood can apply (see Frühwirth-Schnatter, 2006, p.53). Since large \( N \) is not particularly a strength in the samples used for the present study, we consider the more robust alternative of Bayesian estimation using Markov chain Monte Carlo (MCMC) simulations.

The use of Bayesian techniques is convenient in that it allows us to make use of priors which introduce a smoothing effect in the switching likelihood function and reduce the likelihood of obtaining spurious modes in cases where the EM algorithm would lead to degenerate solutions (Frühwirth-Schnatter, 2006). Additionally we gain the ability to assess the degree of parameter uncertainty; by simply interrogating the available posterior distribution and, more importantly, reduce the reliance on asymptotic normality. Furthermore, it yields valid inference even when certain regularity conditions such as; small sample size, small component weights and small degrees of freedom are violated.\(^{15}\)

To estimate the parameter space \( \vartheta \) using Bayesian techniques, we impose specific informative priors on the parameters and then invoke Bayes theorem which combines information about \( \vartheta \) contained in the complete data likelihood \( p(y, S | \vartheta) \), with the prior information contained in the prior distribution \( p(\vartheta) \) to obtain the complete posterior distribution \( p(\vartheta | y, S) \) thus;

\[
p(\vartheta | y, S) \propto p(y, S | \vartheta)p(\vartheta).
\]

The choice of the prior distribution determines whether or not the posterior distribution is proper or improper.\(^{16}\) To ensure that the posterior distribution is proper, one has to choose

\(^{14}\)C.-J. Kim and Nelson (1999b) provide an excellent general level introduction of the EM algorithm in the context of state-space Gaussian Markov-switching models.

\(^{15}\)In the early days of Markov mixture modelling, the method of moments technique was popular until the advent of powerful modern computing technology (see Quandt & Ramsey, 1978). Other evolving approaches for estimating Markov mixture models are; dynamic programming (I. Kim, 1993), distance based methods, including the penalized minimum distance based estimation (J. Chen & Kalbfleisch, 1996), and Bayesian distance based estimation (Hurn, Justel, & Robert, 2003; Sahu & Cheng, 2003).

\(^{16}\)The process of choosing priors for a Markov mixture model is not a trivial decision, especially because improper priors often lead to improper posteriors, since they are often not integrable over the parameter space. One possible solution is to use hierarchical priors or partially proper priors (see Frühwirth-Schnatter, 2006, p.61),
proper prior densities and assume that the density of the prior distribution is independent of
the distribution of the transition probabilities, thus;

\[ p(\theta) = \prod_{k=1}^{K} p(\theta_k)p(\xi), \quad (14) \]

in which case, the posterior is given as;

\[ p(\theta|y, S) = \prod_{k=1}^{K} p(\theta_k|y, S), p(\xi|S), \quad (15) \]

so that the posterior of the complete data Bayesian parameter estimation is the product of the
posteriors from the one-step ahead predictive densities for each state \( k = \{1, 2\}, \)

\[ p(\theta_k|y, S) \propto \prod_{t:S_t=k} p(y_t|\theta_k, y_-1)p(\theta_k), \quad (16) \]

and the posterior of the transition probabilities

\[ p(\xi|S) \propto p(S_0|\xi) \prod_{j=1}^{K} \prod_{k=2}^{K} \xi_j^{N_{jk}(S)} p(\xi), \quad (17) \]

where \( N_{jk}(S) = \# \{ S_{t-1} = j, S_t = k \} \) counts the number of transitions from state \( j \) to \( k \).

4.2 Hierarchical priors and the choice of hyperparameters

For Markov mixture models, it is not always possible to choose simple conjugate priors for
the mixture likelihood \( p(y|\theta) \). A conjugate analysis is however possible for the complete-data
likelihood, \( p(y, S|\theta) \), if one can carefully choose the hyperparameters of the subjective priors.
Because the results from Bayesian analysis with subjective priors are highly sensitive to the
choice of hyperparameters, we suppress this sensitivity by using hierarchical priors. In this
case, the hyperparameters \( \delta \) are treated as unknown quantities with the choice of the prior
distribution \( p(\delta) \) being\(^{17}\)

\[ p(\theta_1, \ldots, \theta_K, \delta) = \prod_{k=1}^{K} p(\theta_k|\delta), \quad (18) \]

\(^{17}\)This approach is particularly useful when Bayesian parameter estimation is based on data augmentation,
the main advantage being the fact that these priors are invariant to relabelling in the states of the Markov
chain (see Frühwirth-Schnatter, 2006, p.335). Recent examples of application of hierarchical priors include; in
Kaufmann (2002, 2000) for Markov mixture models of the business cycle, and Richardson and Green (1997) for
univariate normal mixtures.
which renders all the parameters in all the states dependent apriori,\textsuperscript{18} but maintains the property that the state-specific parameters are independent of the transition matrix, that is \( p(\theta) = p(\xi)p(\theta_1, \ldots, \theta_K) \).

The choice of the prior distribution for the transition matrix \( \xi \) follows a Dirichlet distribution;

\[
\xi_k \sim D(e_{k1}, \ldots, e_{kK}), \quad k = 1, \ldots, K, \tag{19}
\]

which is necessary to preserve the assumption that each row of \( \xi \) is independent apriori and is a discreet probability distribution.\textsuperscript{19} For the other parameters in the model, we choose invariant hierarchical independence priors, as in Richardson and Green (1997), with the hyperparameters selected according to Frühwirth-Schnatter (2006, p.185-87). In particular, the regression coefficients, \( \beta_k \sim N_d(b_{0,k}, B_{0,k}) \), are assumed to be normally distributed. Furthermore, the regression variances, \( \sigma^2_{\epsilon, k} \sim \mathcal{G}^{-1}(c_{0,k}, C_{0,k}) \), have an Inverse Gamma distribution, with the hyperparameter \( C_0 \) being a random variable with a Gamma prior of its own i.e., \( C_0 \sim \mathcal{G}(g_0, C_0, G_{0,k}) \). The chosen values are as follows;

\[
\begin{align*}
b_{0,k} &= \bar{y}, \quad B_{0,k} = \text{Diag}(B_{0,1}, \ldots, B_{0,d}); \\
c_{0,k} &= v_c, \quad g_0 = 0.5, \quad G_0 = g_0\phi(v_c - 1)s^2_y;
\end{align*} \tag{20}
\]

where \( B_{0,k} = 0.25 \) if \( B_{k,j} \) is the switching AR coefficient, and \( B_{0,k} = 10 \) for other coefficients. \( b_{0,k} = \bar{y} \) is the sample mean of the dependent variable, \( s^2_y \) is the sample variance of the dependent variable, while the location and scale parameters for the gamma distribution are given as \( v_c = 2.5 \), and \( \phi = 0.5 \) respectively.

4.3 MCMC sampling schemes; multi-move, and unconstrained permutation sampling

The nature of the posterior distributions derived from Markov mixture models have repercussions on the performance of standard (Gibbs and Metropolis-Hastings(M-H)) sampling techniques via MCMC methods, as demonstrated in Frühwirth-Schnatter (2001b, 2001a). This is because knowledge of which state, \( (k = 1, 2) \), the sampled parameters correspond to is unavailable and thus label switching may have occurred.\textsuperscript{20} Consequently, it is worth considering that any MCMC method used to sample from a Markov mixture posterior should account in some way for this Bayesian unidentifiability problem. Interestingly, earlier applications of Markov mixture models, as in Albert and Chib (1993), C.-J. Kim and Nelson (1999a), and Shephard (1994) have used

\textsuperscript{18}Notice that as a result of this dependence, \( p(\theta_1, \ldots, \theta_K) = \int p(\theta_1, \ldots, \theta_K|\delta)p(\delta)d\delta \neq \prod_{k=1}^K p(\theta_k) \).

\textsuperscript{19}The Dirichlet distribution helps to model the prior belief that the probability of persistence \( \xi_{kk} \) (of remaining in a particular policy state), is greater than the probability of transition between states \( \xi_{jk} \) (see Frühwirth-Schnatter, 2001b). The common use in the literature of the beta distribution is problematic because it does not explicitly allow for the modelling of different degrees of persistence between state transition and state retention.

\textsuperscript{20}The term label switching was first introduced by Redner and Walker (1984) to describe the invariance of the mixture likelihood under relabeling of the states. This simply means that in the course of sampling from the mixture posterior distribution, the labelling of the unobserved states changes.
sampling methods such as; Gibbs, Metropolis-Hastings or related schemes that have neglected the label switching problem, which was latter addressed using various modifications in Chib (1996), Stephens (2000), Frühwirth-Schnatter (2001b), and Celeux, Hurn, and Robert (2000).  

Here we use the unconstrained permutation sampling scheme, fully demonstrated in Frühwirth-Schnatter (2001b). Its major strenght lies in its ability to mix well over the mixture posterior and to evenly explore the entire distribution of the parameter space by jumping between the various labels of the states in a balanced way without getting trapped at one modal region. The scheme involves data augmentation by estimating the augmented parameters $(S, \theta)$ through sampling from the posterior distribution $p(S, \theta | y) \propto p(y | S, \theta)p(S | \theta)p(\theta)$, where on the one hand, $\theta$ is sampled conditional on knowing the states $S$, and on the other hand, $S$ is sampled conditional on knowing $\theta$. The algorithm for unconstrained permutation sampling of the parameters and states are described in four stages below:

(I). **Phase 1:** Gibbs sampling for the parameters $\theta$

(a). Start with a carefully chosen initial state process $S^{(0)}$ and repeat the steps below for $m = 1, \ldots, M, M_0, \ldots, M + M_0$ iterations.

(b). Sample the transition matrix $\xi$ from the posterior distribution of the transition probabilities, $p(\xi | S^{(m-1)})$.

(c). Sample the model parameters $\theta_1, \ldots, \theta_K$ from the complete-data posterior $p(\theta_1, \ldots, \theta_K | y, S^{(m-1)})$ and store the sampled parameters for this iteration as $\theta^{(m)} = (\theta_1^{(m)}, \ldots, \theta_K^{(m)}, \xi^{(m)})$.

(II). **Phase 2:** Metropolis-Hastings sampling for the ergodic initial transition probability $\eta_{S_0}$

(d). Use M-H algorithm to sample from the joint posterior $p(\xi | S)$, which is the self conjugate to the Dirichlet prior distribution specified above. Thus, starting from the old transition matrix $\xi^{old}$, a new transition matrix $\xi^{new}$ is proposed by drawing some or all the rows of $\xi^{old}$ from the Dirichlet proposal density where the acceptance rate for the M-H is:

$$
\min \left\{ 1, \frac{p(\xi^{new} | S) \prod_{j=1}^{K} g_j(\xi^{new}_j)}{p(\xi^{old} | S) \prod_{j=1}^{K} g_j(\xi^{old}_j)} \right\} = \frac{\eta_{S_0}^{new}}{\eta_{S_0}^{old}}; \quad U \leq \frac{\eta_{S_0}^{new}}{\eta_{S_0}^{old}},
$$

where $\xi^{new}$ is accepted if $U \leq \frac{\eta_{S_0}^{new}}{\eta_{S_0}^{old}}$, and $U$ is a random draw from the uniform distribution $\mathcal{U}[0, 1]$, with $\eta_{S_0}^{new}$ being the initial transition probability.

(III). **Phase 3:** Multi-move sampling of the path for the hidden Markov chain $S^{(m)}$ given $\theta$

21In particular, Chib (1996) suggest the use of multi-move sampling, Robert and Titterington (1998) suggest methods based on reparameterization, whereas Frühwirth-Schnatter (2001b) suggest the use of unconstrained or constrained permutation sampling.

22see Frühwirth-Schnatter (2001b, 2006, p.338-40) for a detailed step by step description.
(e). Run the standard recursive filtering algorithm (which is contained in the Appendix) conditional on \( \vartheta \) and store the filtered state probability distribution \( Pr(S_t = j | y^t, \vartheta) \) for all \( k \) and \( t = 1, \ldots, T \).

(f). Sample \( S_T^{(m)} \) from the filtered state probability distribution \( Pr(S_T = j | y^T, \vartheta) \).

(g). For \( t = T - 1, T - 2, \ldots, 0 \), sample \( S_t^{(m)} \) from the conditional distribution \( Pr(S_t = j | S_{t+1}^{(m)}, y^t, \vartheta) \) which requires knowledge of the filtered state probabilities stored in (e).

(IV). **Phase 4**: Unconstrained (random) permutation sampling to conclude draws for \( \vartheta, \xi \) and \( S^{(m)} \)

(h). Conclude each draw by selecting randomly one of \( K! \) possible permutations \( \rho_s(1), \ldots, \rho_s(K) \) of the current labelling which is then applied to; the transition matrix \( \xi^{(m)} \), the state-specific parameters \( \theta_1^{(m)}, \ldots, \theta_K^{(m)} \), and the states \( S^{(m)} \). In particular, this involves three procedures:

(h1). Each element \( \xi_{jk}^{(m)} \) of the simulated transition matrix is substituted by \( \xi_{\rho_s(j),\rho_s(k)}^{(m)} \) for \( j, k = 1, \ldots, K \)

(h2). The state specific parameter \( \theta_k^{(m)} \) is substituted by \( \theta_{\rho_s(k)}^{(m)} \) for \( k = 1, \ldots, K \)

(h3). The states \( S_t^{(m)} \) are substituted by \( \rho_s\left(S_t^{(m)}\right) \) for \( t = 0, \ldots, T \).

(i). Store the actual values of all states as \( S_t^{(m)} \), increase \( m \) by one and return to step (a) above.

(j). Finally, the first \( M_0 \) draws are discarded. In the present implementation, we discard the first 2000 simulations.

4.4 Specification tests and model diagnostics

To test the proposition that commodity price slacks are relevant ingredients in the policy formulation processes, we use the Bayes factor to make comparisons between alternative specifications of the model, see Kass and Raftery (1995) for an authoritative review. The odds ratio in favour of a model, say \( \mathcal{M}_1 \) is given as;

\[
\frac{p(\mathcal{M}_1 | y)}{p(\mathcal{M}_2 | y)} = \frac{p(y | \mathcal{M}_1) p(\mathcal{M}_1)}{p(y | \mathcal{M}_2) p(\mathcal{M}_2)}
\]  

(21)

\( \text{The algorithm for filtering and smoothing the state probabilities are contained in the Appendix including the derivation of the forward-filtering-backward smoothing schemes.} \)

\( \text{This is based on the common knowledge that the regularity conditions for justifying the use of the } \chi^2 \text{ approximation of the likelihood ratio statistic in classical maximum likelihood do not hold in this context, because the state-dependent parameters are unidentified under the alternative hypothesis that there are regime changes.} \)
where in words, the terms will read \([\text{posterior odds} = \text{Bayes factor} \times \text{prior odds}]\), and the ratio of the marginal likelihoods \(B_{12} = p(\mathbf{y} | \mathcal{M}_1) / p(\mathbf{y} | \mathcal{M}_2)\) is the Bayes factor.\(^{25}\) The interpretation of the statistic follows the recommendation by Jeffrey (1961, p. 432), where values between 1 to 3.2 indicate weak evidence in favour of \(\mathcal{M}_1\), values between 3.2 and 10 indicate substantial evidence, values between 10 and 100 indicate strong evidence and values above 100 indicate decisive evidence in favour of \(\mathcal{M}_1\) (see Kass & Raftery, 1995).\(^{26}\)

For a Markov mixture model, the marginal likelihood \(p(\mathbf{y} | \mathcal{M}_k)\) is not available in closed form, and obtaining good numerical approximations is a non-trivial integration problem. Popular methods used to estimate marginal likelihoods in the literature include: the Chib’s estimator (Chib, 1995; Chib & Jeliazkov, 2001), importance sampling based on mixture approximations (Geweke, 1989; Frühwirth-Schnatter, 1995), and bridge sampling (Meng & Wong, 1996; Frühwirth-Schnatter, 2004) amongst others. Han and Carlin (2001) and Frühwirth-Schnatter (2001a) provide comparative reviews, which indicate that the bridge sampling technique for estimating the marginal likelihood function is optimal in the case of Markov mixture models. Hence, in the present study we use the bridge sampling technique to compute the marginal likelihoods which are then used to obtain the Bayes factor for model comparison.

To apply the bridge sampler described and investigated in detail by Frühwirth-Schnatter (2001a, 2004, 2006, p.150-52), we require an arbitrary function \(\alpha(\mathbf{\theta}_K)\) such that,

\[
C_\alpha = \int \alpha(\mathbf{\theta}_K)p(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K)q(\mathbf{\theta}_K)d\mathbf{\theta} > 0,
\]

where \(q(\mathbf{\theta}_K)\) is a distribution with a known normalizing constant and density (often called the importance density) that approximates the posterior distribution of the parameters. Bridge sampling of the marginal likelihood is then based on the following:

\[
1 = \frac{\int \alpha(\mathbf{\theta}_K)p(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K)q(\mathbf{\theta}_K)d\mathbf{\theta}}{\int \alpha(\mathbf{\theta}_K)q(\mathbf{\theta}_K)p(\mathbf{\theta}_K | \mathcal{M}_K)d\mathbf{\theta}} = \frac{\mathbb{E}_q(\alpha(\mathbf{\theta}_K)p(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K))}{\mathbb{E}_p(\alpha(\mathbf{\theta}_K)q(\mathbf{\theta}_K))}.
\]

Let \(\mathbb{E}_f(h(\mathbf{\theta}_K))\) denote the expectation of \(h(\mathbf{\theta}_K)\) with respect to the density \(f(\mathbf{\theta}_K)\), then by substituting \(p(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K) = p^*(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K)/p(\mathbf{y} | \mathcal{M}_K)\), where \(p^*(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K) = p(\mathbf{y} | \mathbf{\theta}_K)p(\mathbf{\theta}_K)\) is the non-normalized posterior, we get the bridge sampling identity thus;

\[
p(\mathbf{y} | \mathcal{M}_K) = \frac{\mathbb{E}_q(\alpha(\mathbf{\theta}_K)p^*(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K))}{\mathbb{E}_p(\alpha(\mathbf{\theta}_K)q(\mathbf{\theta}_K))},
\]

for practical purposes, the expectations in (24) are substituted by sample averages. In particular, the numerator which is an expectation with respect to the mixture posterior density \(p^*(\mathbf{\theta}_K | \mathbf{y}, \mathcal{M}_K)\) is approximated using MCMC draws \(\{\mathbf{\theta}_K^{(m)}, m = 1, \ldots, M\}\), whereas the denom-

\(^{25}\)To focus on the Bayes factor we follow the common practice of attaching equal prior weights to models 1 and 2 so that the prior odds ratio becomes one, i.e., \(p(\mathcal{M}_1)/p(\mathcal{M}_2) = 1\), and hence redundant.

\(^{26}\)It should be noted that Jeffrey (1961)’s values are just as arbitrary as the frequentists’ choices of 1, 5 and 10 per cent levels of significance.
inator which is an expectation with respect to the importance density, $q(\vartheta_K)$, is approximated using \emph{i.i.d} draws $\{\tilde{\vartheta}_K^{(l)}, l = 1, \ldots, L\}$. Thus, by dropping the theoretical expectations, the empirical representation of the resulting bridge sampling estimator is:

$$\hat{p}(y|M_K) = \frac{L^{-1} \sum_{l=1}^{L} (\alpha(\tilde{\vartheta}_K^{(l)}) p^*(\tilde{\vartheta}_K^{(l)} | y, M_K))}{M^{-1} \sum_{m=1}^{M} (\alpha(\vartheta_K^{(m)}) q(\vartheta_K^{(m)}))}. \quad (25)$$

The implementation of all the procedures described in this section are carried out in MATLAB using the package \texttt{bayesf version 2.0} (see Frühwirth-Schnatter, 2008, and the manual therein), supplemented in some cases with specifically written functions in R.

### 4.5 Data

In general, we consider any economy in which primary commodities make up more than 60 per cent of the total merchandise export as a primary commodity export economy. We study a sample of six economies with varying sample periods including: Brazil (1998Q1-2014Q4), Chile (1995Q1-2014Q4), Canada (1993Q1-2014Q4), Mexico (1990Q1-2014Q4), Nigeria (1991Q1-2014Q4) and South Africa (1990Q1-2014Q4). The number of countries under investigation and the sample lengths are influenced by several factors including; clearly separated and defined roles for monetary and fiscal authorities, regional relevance, data availability, and the fact that these countries switched to interest-rate based monetary policy rules about the early or middle 1990s. Canada is included in the sample as a benchmark for comparison of results between emerging markets and developed markets. Monetary policy variables including; inflation, exchange rate, and the monetary policy rate, are retrieved from the International Financial Statistics (IFS) of the IMF and augmented when necessary with data from the relevant national statistics body.

The commodity price slack variable is country-specific being a price index of the major commodities traded by each country, retrieved from several publications of the IMF’s primary commodity prices.\textsuperscript{27} The slack is measured in terms of deviations of the variable from its long run path in percentage terms. As for the fiscal variables; we use the updated version of the comprehensive estimates of public debt to GDP ratio calculated by Reinhart and Rogoff (2011). Government spending and revenue in GDP are both retrieved from World Development Indicators (WDI), World Bank. The output gap variable is calculated as the HP-filtered divergence between actual output and its trend value in percentage terms, scaled by 100. Annual series are transformed to quarterly series using a cubic spline procedure which assigns each value in the low frequency series to the last high frequency observation associated with the low frequency period, then places all intermediate points on a natural cubic spline. The advantage of this method is that it is a global interpolation and hence is capable of preserving the underling relationships in a regression (see Fox, 2000).

\textsuperscript{27}The country-specific relevant commodities are: for Brazil; iron ore and petroleum, for Chile; copper ore, refined and raw copper, for Canada; crude petroleum, and refined petroleum, for Mexico; crude petroleum, for Nigeria; crude petroleum and petroleum gas, and for South Africa; gold, diamonds and platinum.
5 Bayesian estimation results

5.1 Exploratory Bayesian diagnostics

We start by checking for model specification dominance, using Bayes factor to compare the linear versus Markov switching (non-linear) specification of the estimated monetary and fiscal policy rules. The results for the marginal log likelihoods, and the Bayes factor for the switching versus linear version of the estimated monetary policy rules are presented in Table 3. The results clearly show that the non-linear switching specification dominates the linear specification for all the countries, although the strength of the evidence in favour of the switching regression varies by country. For Mexico, Nigeria, South Africa and Chile, the evidence is strong, whereas for Brazil and Canada the evidence is relatively week, thus we proceed with the discussion of the results for the Markov switching specification.

Convergence, stability, and identifiability of the Bayesian estimates are monitored through the visualization of the MCMC output from the posterior distribution. Figure 1 displays the sampled parameter values for inflation, $\alpha_\pi$ in Equation 4, against the simulations for the monetary policy rule in each country. This is used mainly for convergence analysis, in other words, to determine the appropriate number of iterations on the posterior required to make reasonable inference about the parameters, and the appropriate number of iterations for the burn-in phase after convergence has been achieved. In this case convergence is achieved at

<table>
<thead>
<tr>
<th>Country</th>
<th>Marginal log likelihood</th>
<th>Bayes factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathcal{M}_1$: Switching</td>
<td>$\mathcal{M}_2$: Linear</td>
</tr>
<tr>
<td>Chile</td>
<td>-120.48 (0.003)</td>
<td>-131.76 (0.002)</td>
</tr>
<tr>
<td>Mexico</td>
<td>-266.32 (0.008)</td>
<td>-349.63 (0.001)</td>
</tr>
<tr>
<td>Nigeria</td>
<td>$1.44e^{05}$ (0.008)</td>
<td>-218.78 (0.001)</td>
</tr>
<tr>
<td>Brazil</td>
<td>-134.22 (0.004)</td>
<td>-137.71 (0.002)</td>
</tr>
<tr>
<td>South Africa</td>
<td>-140.79 (0.003)</td>
<td>-153.91 (0.001)</td>
</tr>
<tr>
<td>Canada</td>
<td>-86.75 (0.005)</td>
<td>-88.26 (0.001)</td>
</tr>
</tbody>
</table>

Marginal log likelihoods of the linear and non-linear specifications are computed based on bridge sampling from the complete-data posterior. Standard errors are in parenthesis. Results are based on 12,000 simulations with a burn-in of 2000 draws. Bayes factor is the ratio of the Marginal log likelihood; where values between 1 to 3.2 indicate weak evidence in favour of $\mathcal{M}_1$, values between 3.2 and 10 indicate substantial evidence, values between 10 and 100 indicate strong evidence and values above 100 indicate decisive evidence in favour of $\mathcal{M}_1$ with symmetric negative values indicating various degrees of support for $\mathcal{M}_2$. 
different levels of iteration with Chile having the highest of about 11,000 after a burn-in of 1000 and Brazil being somewhere around 4,900 iterations. We observe that the sampler discriminates well between the two states for the values of the inflation coefficient, the only exceptions being Chile and Brazil where the discrimination between the two states seems to be fuzzy.

The correct identification of the number of states can be inferred from the visualization of the scatter plots of the point process representation of the MCMC draws for the parameters. This can help to answer questions about whether the model is over-fitting in the number of states without worrying about the issue of label switching (see Kaufmann, 2002, for example). Figure 2 is a plot of the point process representation of the parameters for Chile, which typically depicts the average pattern observed across the sample of countries as a whole. The MCMC draws are expected to scatter around the points corresponding to the true point process representation with the spread of the clouds representing the uncertainty of estimating the points. Further, the number of clusters visible should reveal the number of states present and which parameters are driving the switching process. The first four panels in Figure 2 indicate that the most significant drivers for switching behaviour are the sampled values for the inflation parameter. This finding provides further empirical support to Leeper (1991)’s classification of policy stance based on the reaction of policy to inflation. In addition, since at least two clusters are vaguely
discernible from the panels in the figure, there is an implication that the model does not suffer from problems of over or under-fitting of the states. The spread of the clusters however suggests modest degrees of uncertainty and wider confidence intervals for some parameter estimates.

5.2 The monetary policy reaction function

The Bayesian results for the Markov-switching monetary policy rule (Equation 4) are in Table 4. The ergodic means are estimates of the averages from the posterior distribution after identification using unconstrained permutation sampling of 10,000 iterations after a burn-in phase of 2,000 iterations. The 95 percent credible set contains the 95 percent highest posterior density interval (HPDI) for the estimated parameters. $\pi^*$ is the implied inflation target, $P_{ii}$ is the persistence probability, or the probability of remaining in the same state when moving from periods 1 to 2. $P_{ij}$ is the transition probability and $E(D)$ is the expected duration of the policy rule. Apart from Brazil the results show a clear distinction between the “active” and “passive” monetary
policy regimes for Chile, Canada, Mexico, Nigeria and South Africa. The classification of the policy regime follows Davig and Leeper (2006, 2011)'s characterization. That is, estimates of the inflation parameter which satisfy the Taylor principle, $\alpha_\pi > 1$, are regarded as active monetary policy regimes, whereas values that are less than one are classified as passive monetary policy regimes.

The results show that during active monetary policy regimes, the reaction of interest rates to inflation ranges between 3.15 and 1.23. These estimates are within close range when compared to recent results in the literature. For example, in a constant-regime Bayesian IV estimation of the policy rule for Canada Lubik and Schorfheide (2007) obtain a value of 1.51 compared to the present 1.23 obtained on the inflation coefficient in the active regime. Similarly, for Chile; we obtain 2.93 and 0.11, Mexico; 1.47 and 0.01, South Africa; 2.51 and 0.04, and Nigeria; 3.15 and -0.18 for the active and passive policy regimes respectively. These results are somewhere in-between single regime policy rules estimated for these economies by Mohanty and Klau (2005). \[^{28}\] We obtain the implied inflation target for each of the economies in the sample by plugging in the inflation coefficients during the active monetary policy regime into Equation 6. The implied inflation target when compared with the actual target announced by the respective central banks indicates that the central bank’s reaction to inflation is often not “hawkish” (aggressive) enough to achieve the announced target. For example, during the period under investigation, Chile and Mexico announced a target of 3 percent, but the policy rule followed implies a target value of about 4 percent and 8.1 percent respectively. Canada announced 2 percent, the rule implies 2.52 percent. Nigeria and South Africa announced targets of between 6 to 9 percent and 3 to 6 percent respectively, though the estimated policy rules suggest they are pursuing an inflation target of around 11.5 and 8.7 percent respectively.

In so far as output stabilization is concerned, the evidence is mixed. In most cases, we get the expected positive sign on the output gap especially in the active policy regime. During passive monetary regimes, the phenomenon noted by Owyang and Ramey (2004), where output stabilization receives greater weights relative to inflation stabilisation, does not seem to hold in these economies, except for Canada. Here we caution that the estimates for the output gap should be interpreted with care, the reason being that they are likely to suffer from a downward bias due to the greater difficulty of measuring the output gap for emerging economies compared to industrialized economies, which basically stems from the greater importance of supply side shocks in output dynamics for these class of economies. Another potential reason for the mixed evidence on output stabilization could be the role of other macroeconomic policies, especially fiscal policy, in output stabilization for emerging market economies (see Sidaoui, 2003, for the experience in Mexico).

In all the countries, except for Chile and Nigeria, the exchange rate variable uniformly returns positive average coefficients, suggesting that monetary authorities ‘lean against the wind’ by tightening monetary policy in response to nominal exchange rate depreciations and

\[^{28}\]In particular, Mohanty and Klau’s results on the inflation coefficient are; Chile 0.97, Mexico 0.55, and South Africa 0.08.
<table>
<thead>
<tr>
<th></th>
<th>Chile</th>
<th>Mexico</th>
<th>Nigeria</th>
<th>Brazil</th>
<th>South Africa</th>
<th>Canada</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Active policy</td>
<td>Passive policy</td>
<td>Active policy</td>
<td>Passive policy</td>
<td>Active policy</td>
<td>Passive policy</td>
</tr>
<tr>
<td></td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>12.347 [10.269, 14.205]</td>
<td>0.902 [0.566, 1.038]</td>
<td>12.56 [12.531, 12.59]</td>
<td>0.1015</td>
<td>0.014</td>
<td>12.56 [12.54, 12.58]</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>2.935 [2.115, 3.75]</td>
<td>0.118 [0.001, 0.226]</td>
<td>1.471 [1.261, 1.680]</td>
<td>0.0104</td>
<td>0.0103</td>
<td>0.1015</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>0.515 [0.345, 0.668]</td>
<td>0.008 [-0.044, 0.060]</td>
<td>1.013 [0.849, 1.177]</td>
<td>0.882</td>
<td>0.058</td>
<td>0.849</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>-0.028 [-0.038, -0.018]</td>
<td>0.002 [0, 0.004]</td>
<td>2.464 [2.165, 2.762]</td>
<td>0.055</td>
<td>0.239</td>
<td>0.292</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>1.196 [1.188, 1.208]</td>
<td>0.018 [0.01, 0.026]</td>
<td>0.144 [0.101, 0.187]</td>
<td>0.021</td>
<td>0.002</td>
<td>0.101</td>
</tr>
<tr>
<td>$a_0(s_M^t)$</td>
<td>-0.445 [-0.691, -0.199]</td>
<td>0.827 [0.751, 0.903]</td>
<td>0.753 [0.723, 0.784]</td>
<td>0.551</td>
<td>0.822</td>
<td>0.881</td>
</tr>
<tr>
<td>$a_9(s_M^t)$</td>
<td>3.92</td>
<td>6.14</td>
<td>1.37</td>
<td>-224.49</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_9(s_M^t)$</td>
<td>0.95</td>
<td>0.69</td>
<td>0.28</td>
<td>0.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_9(s_M^t)$</td>
<td>0.05</td>
<td>0.31</td>
<td>3.57</td>
<td>14.2</td>
<td></td>
<td></td>
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<tr>
<td>$\pi_1(s_M^t)$</td>
<td>3.22</td>
<td>0.036</td>
<td>0.033</td>
<td>0.018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_1(s_M^t)$</td>
<td>-0.182 [-0.218, -0.142]</td>
<td>-0.182 [-0.218, -0.142]</td>
<td>-0.934 [-1.21, -0.686]</td>
<td>0.018</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_1(s_M^t)$</td>
<td>0.426 [0.345, 0.668]</td>
<td>0.008 [-0.044, 0.060]</td>
<td>1.013 [0.849, 1.177]</td>
<td>0.882</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi_1(s_M^t)$</td>
<td>-0.028 [-0.038, -0.018]</td>
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<td></td>
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<td>0.753 [0.723, 0.784]</td>
<td>0.551</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The ergodic means are estimates of the averages from the posterior distribution after identification using unconstrained permutation sampling of 10,000 iterations with a burn-in phase of 2,000 iterations. The 95% credible set contains the 95% highest posterior density interval (HPDI) for the estimated parameters. Standard errors are in parentheses. $\pi^*$ is the implied inflation, $P_{ij}$ is the persistence probability, i.e. the probability of remaining in the same state from periods 1 to 2. $\pi_{ij}$ is the transition probability, and $E(D)$ is the expected duration of the policy rule.
vice versa, the degree of tightening, depending on the current state (active or passive) of the policy regime. In particular, the range of the coefficient is between 2.46 percent (during active policy regimes in Mexico), and 0.01 percent (during passive policy regimes in South Africa), of the initial percentage-point exchange rate movement. The results are slightly higher than single regime estimates by Mohanty and Klau (2005) for emerging economies and generally lower than the Bayesian IV estimates by Lubik and Schorfheide (2007) for industrialised economies, but loosely comparable to GMM estimates by Clarida et al. (1998) for industrialized economies. The results further show that there is a high degree of interest rate smoothing by the central banks, this can be inferred from the average estimate of the autoregressive parameter which is somewhere around 0.65, depending on the prevailing policy regime.

Figure 3: Filtered and smoothed transition probabilities for active monetary policy

The stationary transition probabilities for the active policy regime $P_{ii}$ range between 0.95 in Nigeria to 0.68 in Canada whereas for the passive regime $P_{jj}$ it is between 0.97 in South Africa and 0.74 in Brazil. The high degree of persistence in most countries indicates that once policy is in a certain regime, the likelihood of remaining in the same regime the next period is very high. Given the persistence of the policy regimes, the results imply that the average expected duration $E(D)$ of an active policy episode is around 8 quarters (2 years) and that of a passive
episode is around 4 quarters (1 year). The filtered (plotted as bars) and smoothed (plotted as lines) transition probabilities of being in the active monetary policy regime are depicted in Figure 3.\textsuperscript{29} This plot helps us to answer questions about whether the periods estimated to be active and passive, correspond with the narrative and observed accounts of policy history in these economies. The cases of Brazil, Chile and South Africa seem to stand out, apart from the short spells of passive monetary policy regimes in the early 2000s monetary policy has mostly been active from 2004 and beyond. These identified periods of policy episodes are easily comparable in the sense that they closely match the periods of policy changes identified in the narrative accounts of policy history in Latin America by Langhammer and de Souza (2005), in South Africa by Aron and Muellbauer (2007), and in Canada by Lubik and Schorfheide (2007).

5.3 The fiscal policy reaction function and policy interactions

For the fiscal policy reaction function we begin by discussing the results for the test of the dominant specification between the linear and Markov switching (non-linear) specifications. The marginal log likelihoods and Bayes factor for both specifications of the fiscal policy rule are presented in Table 5. With the exception of Brazil, the Bayes factor indicates that the non-linear

<table>
<thead>
<tr>
<th>Country</th>
<th>Marginal log likelihood</th>
<th>Bayes factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\mathcal{M}_1): Switching</td>
<td>(\mathcal{M}_2): Linear</td>
</tr>
<tr>
<td>Chile</td>
<td>-44.98</td>
<td>-67.03</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.47e^03</td>
<td>-105.99</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Nigeria</td>
<td>-218.07</td>
<td>-218.58</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Brazil</td>
<td>-49.10</td>
<td>-46.83</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>South Africa</td>
<td>-38.61</td>
<td>-76.47</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Canada</td>
<td>7.69</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.001)</td>
</tr>
</tbody>
</table>

Marginal log likelihoods of the linear and non-linear specifications are computed based on bridge sampling from the complete-data posterior. Standard errors are in parenthesis. Results are based on 12,000 simulations with a burn-in of 2000 draws. Bayes factor is the ratio of the Marginal log likelihood; where values between 1 to 3.2 indicate weak evidence in favour of \(\mathcal{M}_1\), values between 3.2 and 10 indicate substantial evidence, values between 10 and 100 indicate strong evidence and values above 100 indicate decisive evidence in favour of \(\mathcal{M}_1\) with symmetric negative values indicating various degrees of support for \(\mathcal{M}_2\).

Markov switching specification of the fiscal policy rule is the more probable of the two models

\textsuperscript{29}Note that the transition probabilities for the passive monetary policy regime could be conceived as the flip version of Figure 3 by the law of total probability.
considered. The evidence in favour of the non-linear specification is decisive for Mexico, strong for Canada, Chile and South Africa, weak for Nigeria and outrightly rejected for Brazil.

In the same way as for monetary policy, the convergence, stability and identifiability of the Bayesian estimates are monitored through the visualization of the MCMC output from the posterior distribution. The plots indicate that for most of the parameters, the MCMC simulations achieved convergence after about 7,000 simulations with an average acceptance rate of 54 percent. Generally the sampler also discriminates well between the two states, especially for the lagged debt coefficient, the only exception being for Brazil.\(^{30}\)

The Bayesian results for the Markov-switching fiscal policy rule are presented in Table 6. We find that the critical factor distinguishing the two regimes is driven by the sign of the coefficient on past debt \(\gamma_{b_{t-1}}\), the only exceptions being for Nigeria and Brazil (although it is possible to arrive at the same conclusion when one considers the 95% credible set of the estimates for Brazil). Following the characterization in Leeper (1991), when the fiscal authority is constrained by both consumer optimization and an active monetary authority, it must generate sufficient revenue to balance the budget. Hence, whenever the government sufficiently increases (tax) revenue in response to the past level of government debt, i.e., \(\gamma_{b_{t-1}} > 0\), fiscal policy is described as being passive.\(^{31}\) On the other hand, when the fiscal authority is not constrained by the current budgetary conditions it follows an active policy regime, whereby it responds negatively to debt. The active policy regime is more consistent with the predictions of business cycle theory, where fluctuations in revenues and debt are expected to covary negatively (see Davig & Leeper, 2006).

Surprisingly, with the exception of South Africa, the response of fiscal authorities to the output gap does not change across regimes for nearly all of the countries under investigation. This is in contrast with the sign switching estimates for the US by Favero and Monacelli (2005), but similar to the fixed sign results for the US by Davig and Leeper (2006) and the mixed results for EU countries by Galí and Perotti (2003).\(^{32}\) Therefore, we conclude that for the type of economies we investigate here, the response of fiscal policy to the output gap has been consistent in spite of the policy regime. In particular, the results show that fiscal policy has been mostly pro-cyclical in Chile and Nigeria and has been mostly counter-cyclical in Mexico, Brazil, and Canada; but switching between countercyclicality and procyclicality in South Africa depending on the prevailing policy regime in place.

Figure 4 displays the filtered (plotted as bars) and smoothed (plotted as lines) transition probability of the active fiscal policy regime. The degree of persistence in the policy regimes are discernible from the respective stationary transition probabilities. The results clearly indicate

\(^{30}\)MCMC visualization of the parameters against simulation and the point process matrix of the parameters for the fiscal policy rule are available from the authors upon request.

\(^{31}\)Note that in an optimization based framework, it is further required that the response of tax policy to debt be greater than the quarterly real interest rate (see Leeper, 1991; Davig & Leeper, 2006; Chung et al., 2007).

Table 6: Bayesian estimates for regime-switching fiscal policy rule

<table>
<thead>
<tr>
<th>Country</th>
<th>Active policy</th>
<th>Passive policy</th>
<th>Active policy</th>
<th>Passive policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Japan</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
</tr>
<tr>
<td>γ0 (s_i^k)</td>
<td>-0.47</td>
<td>[-0.61, -0.33]</td>
<td>0.47</td>
<td>[0.33, 0.61]</td>
</tr>
<tr>
<td>γ1 (s_i^k)</td>
<td>0.89</td>
<td>[0.78, 0.99]</td>
<td>1.16</td>
<td>[1.06, 1.26]</td>
</tr>
<tr>
<td>γ2 (s_i^k)</td>
<td>-0.23</td>
<td>[-0.32, -0.14]</td>
<td>0.23</td>
<td>[0.14, 0.32]</td>
</tr>
<tr>
<td>σ (s_i^k)</td>
<td>0.59</td>
<td>[0.49, 0.70]</td>
<td>0.91</td>
<td>[0.80, 1.02]</td>
</tr>
<tr>
<td>P_α</td>
<td>0.84</td>
<td>[0.78, 0.90]</td>
<td>0.94</td>
<td>[0.88, 0.99]</td>
</tr>
<tr>
<td>P_ξ</td>
<td>0.15</td>
<td>[0.12, 0.18]</td>
<td>0.05</td>
<td>[0.03, 0.08]</td>
</tr>
<tr>
<td>E(D)</td>
<td>2.06</td>
<td>[1.59, 2.53]</td>
<td>3.04</td>
<td>[2.57, 3.51]</td>
</tr>
<tr>
<td>Norway</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
<td>Ergodic mean</td>
<td>95% Credible set</td>
</tr>
<tr>
<td>γ0 (s_i^k)</td>
<td>0.36</td>
<td>[0.28, 0.44]</td>
<td>0.46</td>
<td>[0.39, 0.54]</td>
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<tr>
<td>γ1 (s_i^k)</td>
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<td>[-0.32, -0.12]</td>
<td>-0.23</td>
<td>[-0.32, -0.14]</td>
</tr>
<tr>
<td>γ2 (s_i^k)</td>
<td>0.23</td>
<td>[0.14, 0.32]</td>
<td>0.23</td>
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<td>[0.12, 0.18]</td>
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<td>[0.03, 0.08]</td>
</tr>
<tr>
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<td>3.04</td>
<td>[2.57, 3.51]</td>
</tr>
</tbody>
</table>

The ergodic means are estimates of the averages from the posterior distribution after identification using unconstrained permutation sampling of 10,000 iterations with a burn-in phase of 2,000 iterations. The 95% credible set contains the 95% highest posterior density interval (HPDI) for the estimated parameters. Standard errors are in parenthesis. P_ξ is the persistence probability or the probability of remaining in the same state from periods 1 to 2. P_ξ is the transition probability and E(D) is the expected duration of the policy rule.
that the policy regimes are persistent. The expected duration of the active fiscal policy regime \(\mathbb{E}(D)\) ranges between 33 quarters in South Africa, and 6 quarters in Brazil. Further, we observe that the fiscal policy episodes in Canada are clearly distinct. Between 2001 and 2006 fiscal policy was mostly active, and since the financial crisis in 2007 until 2014, fiscal policy has been passive in Canada.

The equilibrium outcome of the interactions between monetary and fiscal policy regimes for each country are obtained by taking the joint transition probability matrix, which is a Kronecker multiplication of the individual transition probabilities thus; \(\mathcal{P} = \mathcal{P}_M \otimes \mathcal{P}_F\). The dynamic phases in the equilibrium outcome of the interaction between monetary and fiscal policy regimes are plotted in Figure 5. The joint policy framework is regraded as being “explosive” whenever both monetary and fiscal policy regimes are active, (the red segment in Figure 5), when active monetary policy interacts with passive fiscal policy we obtain periods of “Ricardian” policy (blue segment) when passive monetary policy interacts with active monetary policy, this corresponds to the fiscal theory of the price level “FTPL” (yellow segment) and lastly, when both monetary and fiscal policy are passive, the joint policy interaction is “indeterminate” (white segment).

The figure indicates that for Chile and Mexico, it is the fiscal theory of the price level
Figure 5: Identified equilibrium outcome from joint policy interactions
that has dominated the historical narrative, that is, fiscal policy has mostly been active with monetary policy accommodating it. The results for Nigeria shows that they have been many spells of short visits to all the equilibrium states, with FTPL dominating the historical narrative. In Brazil and South Africa, the dominant outcome of policy interaction is mostly ‘active-active’ and hence explosive policy. The experience in Canada is quite different from the rest as the outcome of the policy mix tends to indicate mostly Ricardian episodes, followed by sustained periods of policy explosiveness.

5.4 Do monetary and fiscal authorities respond to commodity price slacks?

To answer the question of whether or not, and to what extent, monetary and fiscal authorities in resource export economies respond to commodity price slacks, we formally test the hypothesis; $M_1: \alpha_{cm} \neq 0$, against the alternative $M_2: \alpha_{cm} = 0$, by computing the marginal log likelihoods and Bayes factor for each model. The results for the formal test of the relevance of commodity price slacks in the monetary policy rule are presented in Table 7. The results indicate that there is substantial evidence that monetary authorities in Brazil, Canada, Mexico and South Africa react to commodity price slacks in the monetary policy rule, with the Bayes factor in favour of the model with the commodity price slack ranging between 7.76 and 5.17. The evidence is weaker for Chile with a Bayes factor of 0.56, and out-rightly rejected for Nigeria with a negative Bayes factor.33

While we find evidence that the monetary authorities in these resource rich economies respond to commodity price slacks, the results for $\alpha_{cm}$ in Table 4 shows that there is considerable variation by country in the nature of the responses and this often depends on whether policy is in the active or passive regime. Typically, the effect of a commodity price boom for a resource exporting economy would be a real appreciation in the domestic currency, (this could take the form of a nominal currency appreciation under a floating exchange rate regime or money inflows and hence inflation if the country runs a fixed exchange rate regime), excessive expansion of credit, increase in the level of money supply and hence inflation (see Frankel, 2011).34 Under such a scenario, it would be desirable for the monetary authority to tighten policy just enough to limit excess demand for goods that might otherwise show up in the non-tradable sector (for example in a real estate, and/or asset price bubbles).35 Using this criteria the results show that

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33It is important to note that from the MCMC diagnostics conducted for Nigeria we found evidence of switching components in the variances, yielding the possibility of a different result if we had used an independent switching variance specification.

34The documented evidence on the sensitivity of exchange rates to the commodity price booms of the 2001-08 episode shows that Saudi Arabia and the UAE are typical examples of fixed exchange rate countries where the real appreciation occurred via money inflows and inflation. Chile, Mexico, Norway, Russia, and South Africa are examples of floating exchange rate resource rich countries where the real appreciation appeared in the form of nominal currency appreciation. A different scenario played out in Nigeria where a currency appreciation was not observed during the reference boom, due to the use of partly successful capital controls and sterilized foreign exchange purchases by the central bank (see Frankel, 2011, 2007; Y.-c. Chen & Rogoff, 2003)

35Symmetrically, when commodity prices go down, monetary authorities should ease monetary policy to limit
## Table 7: Specification test for commodity slack in monetary policy rule

<table>
<thead>
<tr>
<th>Country</th>
<th>Marginal log likelihood</th>
<th>Bayes factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mathcal{M}<em>1: \alpha</em>{cm} \neq 0$</td>
<td>$\mathcal{M}<em>2: \alpha</em>{cm} = 0$</td>
</tr>
<tr>
<td>Chile</td>
<td>-120.48 (-0.003)</td>
<td>-121.04 (-0.004)</td>
</tr>
<tr>
<td>Mexico</td>
<td>-266.32 (+0.008)</td>
<td>-273.57 (+0.004)</td>
</tr>
<tr>
<td>Nigeria</td>
<td>$1.44e^{03}$ (+0.008)</td>
<td>$1.49e^{03}$ (+0.009)</td>
</tr>
<tr>
<td>Brazil</td>
<td>-134.22 (+0.004)</td>
<td>-140.73 (+0.004)</td>
</tr>
<tr>
<td>South Africa</td>
<td>-140.79 (+0.003)</td>
<td>-145.96 (+0.003)</td>
</tr>
<tr>
<td>Canada</td>
<td>-86.75 (+0.005)</td>
<td>-94.51 (+0.005)</td>
</tr>
</tbody>
</table>

Marginal log likelihoods are computed based on bridge sampling from the complete-data posterior. Standard errors are in parenthesis. Results are based on 12,000 simulations with a burn-in of 2000 draws. Bayes factor is the ratio of the Marginal log likelihoods; where values between 1 to 3.2 indicate weak evidence in favour of $\mathcal{M}_1$, values between 3.2 and 10 indicate substantial evidence, values between 10 and 100 indicate strong evidence and values above 100 indicates decisive evidence in favour of $\mathcal{M}_1$, with symmetric negative values indicating various degrees of support for $\mathcal{M}_2$.

During active monetary policy regimes; Chile, Mexico, South Africa and Canada responded to commodity price booms in a counter-cyclical manner by tightening monetary policy. This is not the case for Brazil and Nigeria where monetary policy seems to be pro-cyclical in response to commodity price booms, thereby accentuating the effects of commodity price fluctuations on the domestic economy. The simultaneous targeting of the headline inflation and international commodity prices by central banks in these resource export economies is akin to the product price targeting (PPT) proposal by Frankel (2011), recommended as the optimal strategy for resource export economies. It is important to note that Canada have been following what may be considered an equivalent strategy whereby policy responds to a monetary conditions index (MCI), which includes domestic prices, the exchange rate and some index of commodity prices (see Lubik & Schorfheide, 2007).

For fiscal policy, the results for the test of the relevance of commodity price slacks in the fiscal rule are presented in Table 8. We formally test the hypothesis; $\mathcal{M}_1: \gamma_{cm} \neq 0$, against the alternative that $\mathcal{M}_2: \gamma_{cm} = 0$ for each country. Overall, the Bayes factor indicates strong evidence in favour of the proposition that $\mathcal{M}_1: \gamma_{cm} \neq 0$. In particular, for Nigeria, the Bayes factor is 14.65; South Africa is 5.68; and Brazil is 3.81, the evidence for Canada is however weaker with a Bayes factor of 2.06. The hypothesis is out-rightly rejected for Mexico. Given that tax receipts are typically endogenous to the business cycle, the effect of commodity prices on the business cycle comes through its indirect impact on government spending. As a result, excess supply for goods and avoid a recession or financial crisis.
fiscal policy tends to be pro-cyclical when government increases spending by more than the proportionate increase in revenue from the commodity price boom. Consequently, if the net impact of government spending is greater than the revenue impact of the commodity price boom, fiscal policy is pro-cyclical with respect to the relevant commodity prices.

The results for the fiscal policy rule are in Table 6. They indicate that fiscal policy has been pro-cyclical in Chile, Nigeria, Brazil and South-Africa but counter-cyclical in Canada. The extent of pro-cyclicality depending on the fiscal regime in place, with the effect being more pronounced mostly during the passive fiscal regimes in most of the countries. These results are generally consistent with the theoretical and empirical literature which provides evidence that developing countries experience larger cyclical fluctuations than their industrialized counterparts in response to commodity price booms. One reason for this lies with the fact that the political and institutional framework to handle these kinds of scenarios are generally weaker in the former (see Frankel, Vegh, & Vuletin, 2013; Kaminsky et al., 2005; Talvi & Vegh, 2005, for examples).

Table 8: Specification test for commodity slack in fiscal policy rule

<table>
<thead>
<tr>
<th>Country</th>
<th>Marginal log likelihood</th>
<th>Bayes factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( M_1 : \gamma_{cm} \neq 0 )</td>
<td>( M_2 : \gamma_{cm} = 0 )</td>
</tr>
<tr>
<td>Chile</td>
<td>-44.98</td>
<td>-55.92</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Mexico</td>
<td>1.47e^{03}</td>
<td>1.51e^{03}</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.015)</td>
</tr>
<tr>
<td>Nigeria</td>
<td>-218.07</td>
<td>-232.72</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>Brazil</td>
<td>-49.10</td>
<td>-52.91</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>South Africa</td>
<td>-38.61</td>
<td>-44.29</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.007)</td>
</tr>
<tr>
<td>Canada</td>
<td>7.69</td>
<td>5.63</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.007)</td>
</tr>
</tbody>
</table>

Marginal log likelihoods are computed based on bridge sampling from the complete-data posterior. Standard errors are in parenthesis. Results are based on 12,000 simulations with a burn-in of 2000 draws. Bayes factor is the ratio of the Marginal log likelihoods; where values between 1 to 3.2 indicate weak evidence in favour of \( M_1 \), values between 3.2 and 10 indicate substantial evidence, values between 10 and 100 indicate strong evidence and values above 100 indicate decisive evidence in favour of \( M_1 \) with symmetric negative values indicating various degrees of support for \( M_2 \).

As a final check, we conduct graphical residual analysis on the estimated monetary and fiscal policy rules. Figure 6 and Figure 7 are plots of the point process representation of the posterior draws of the residuals against draws from a standard normal distribution. Under the null hypothesis the plot of the residual draws against draws from the normal distribution should mimic the normal distribution. Generally, the departures observed are related to the tails of the distribution particularly for Nigeria and Mexico. This suggest that there seems to remain some volatility clustering in some of the residuals and hence a possible improvement.
of the model fit for Nigeria and Mexico, may be to extend the model to allow for independent regime switching variances. In summary, the Markov switching specification seems to fit the data reasonably well, although we observe that some of the residual diagnostics tests suggest under-dispersement, which could be handled through an extension of the Markov switching specification to the variances for selected countries.

6 Conclusion

We have set out to better understand the rules and interactions between monetary and fiscal policy in economies that are characterized by high levels of primary commodities in total merchandise exports. To achieve this we have formally estimated modified versions of monetary and fiscal policy rules in the spirits of Taylor (1993) and Leeper (1991), stylized to account for a commodity price slack component. Further, we have analysed the nature of the interactions between the two policy types by following the framework set out in Davig and Leeper (2006, 2011). To allow for potentially stochastic policy regimes, we endogenize the states of the economy by casting the policy rules in a Markov regime-switching framework before estimating the model parameters using specifically designed Bayesian MCMC sampling techniques, rather than the
The key results from the study are summarized as follows: (i) monetary and fiscal authorities in high resource export economies respond to commodity price slacks, albeit, in different ways depending on the policy regime in place; (ii) monetary and fiscal policy is characterized by distinctive episodes of active and passive policy regimes, driven by the response of monetary policy to inflation and the response of fiscal policy to past government debt; (iii) monetary authorities in these economies often do not act aggressive (‘hawkish’) enough to achieve their announce inflation targets; (iv) there is no evidence of policy synchronization between monetary and fiscal policy in the countries under investigation.

Some qualification on the conclusions are however instructive. Firstly, because the approach used in this paper is based on “simple” monetary and fiscal policy rules, we are not able to make statements about the optimality of the rules estimated. However, the consistent evidence we provide on the relevance of commodity price slacks and the stochastic regime switching nature of the estimated policy rules, holds interesting implications for the correct specification of the monetary-fiscal policy mix in modern workhouse New-Keynesian models used for policy analysis in economies that are characterized by significant export shares of primary commodities.
Appendices

A Algorithm for filtering the state probabilities

Seminal contributions for efficient filtering and smoothing of the unknown states ($S_t$) of a Markov mixture model include Hamilton (1989), and Chib (1996). Our approach here is to describe in a generic way, a unifying algorithm for filtering and smoothing the specific type of Markov switching models used in the study. In particular, algorithms for mixture models that satisfy two basic assumptions: (i) only the present value of the states $S_t$ is allowed to influence the posterior density, and (ii) the state $S_t$ is a first-order Markov chain with arbitrary transition matrix which need not be irreducible or aperiodic and starts from an arbitrary distribution.\(^{36}\)

(A). Filtering the states: To filter the states, we carry out the following steps recursively

1. Obtain the one-step ahead prediction of $S_t$ which serves as the prior

$$Pr(S_t = l|y^{t-1}, \vartheta) = \sum_{k=1}^{K} \xi_{kl}(t-1)Pr(S_{t-1} = k|y^{t-1}, \vartheta)$$

(26)

where $\xi_{kl}(t-1) = Pr(S_t = l|S_{t-1} = k, y^{t-1}, \vartheta)$.

2. Correct the prediction based on information contained in the actual observation $y_t$

$$Pr(S_t = l|y^t, \vartheta) = \frac{p(y_t|S_t = l, y^{t-1}, \vartheta) Pr(S_t = l|y^{t-1}, \vartheta)}{p(y_t|y^{t-1}, \vartheta)}$$

(27)

where $p(y_t|y^{t-1}, \vartheta) = \sum_{k=1}^{K} p(y_t|S_t = k, y^{t-1}, \vartheta) Pr(S_{t-1} = k|y^{t-1}, \vartheta)$. This filter is an adaptive inference tool, and the one step ahead predictive densities $p(y_t|y^{t-1}, \vartheta)$ play the role of the normalizing constant of the non-normalized discreet posterior.

B Algorithm for smoothing the states

After carrying out the filtering, it is possible to conduct time-series post processing by incorporating the whole information set in a backward smoothing algorithm, running from $t = T, T - 1, \ldots, 0$. This backward smoother has been derived independently by Chib (1996) and C.-J. Kim and Nelson (1999a).

(B). Smoothing the states:

\(^{36}\)It is important to note that the description here would still hold for Markov mixture models with homogeneous or time varying transition probabilities as is the case in the application here. The derivations are mostly based on Frühwirth-Schnatter (2006, p.320-326)
(1). The first step is to run the algorithm described in Appendix (A) to obtain the filtered probabilities $Pr(S_t = l|y^t, \vartheta)$ for $l = 1, \ldots, K$, and for each $t = 0, \ldots, T$.

(2). The second step is to run the backward smoother starting from $t = T$ with the distribution $Pr(S_T = l|y, \vartheta)$ and running recursively until $t = 0$.

(3). For each $t = T-1, T-2, \ldots, t_0$, the smoothed probability distribution is derived as

$$Pr(S_t = l|y, \vartheta) = \sum_{k=1}^{K} \xi_{lk}^*(t)Pr(S_t = l|y^t, \vartheta)Pr(S_{t+1} = k|y, \vartheta)$$

where as before, $\xi_{kl}^*(t) = Pr(S_t = k|S_{t-1} = l, y^t, \vartheta)$. To obtain the smoothed probability at a certain point, one only needs to know the filtered probability $Pr(S_t = l|y^t, \vartheta)$, and the smoothed probability distribution one period ahead $Pr(S_{t+1} = l|y, \vartheta)$. When starting with a random initial value, $S_0$, the updated probability statement about the starting value in the light of the observed time series is given as

$$Pr(S_0 = l|y, \vartheta) = \frac{\sum_{k=1}^{K} \xi_{lk}^*(0)Pr(S_0 = l|\xi)}{\sum_{j=1}^{K} \xi_{jkl}^*(0)Pr(S_0 = j|\xi)}$$

where $Pr(S_0 = l|y, \vartheta)$ is the initial distribution.

C Derivation of forward-filtering backward-smoothing

To derive the smoother in Appendix B, we may express the full-sample smoothed probabilities $Pr(S_t = l|y, \vartheta)$ as marginal probabilities, thus;

$$Pr(S_t = l|y, \vartheta) = \sum_{k=1}^{K} Pr(S_t = l, S_{t+1} = k|y, \vartheta) = \sum_{k=1}^{K} Pr(S_t = l|S_{t+1} = k, y, \vartheta)Pr(S_{t+1} = k|y, \vartheta).$$

Under the assumption that the state $S_{t+1}$ is known to be equal to $k$, it is possible to use Bayes’ theorem to obtain these probabilities;

$$Pr(S_t = l|S_{t+1} = k, y, \vartheta) \propto p(y_{t+1}, \ldots, y_T|S_{t} = l, S_{t+1} = k, y^t, \vartheta) \times Pr(S_t = l|S_{t+1} = k, y^t, \vartheta).$$

Note that as $y_{t+1}, \ldots, y_T$ are independent of $S_t$, given $S_{t+1}$, the first term after the proportionality sign cancels out. By applying Bayes’ theorem once more and normalizing, we obtain the full-sample smoothing probability as;

$$Pr(S_t = l|S_{t+1} = k, y, \vartheta) = \frac{\xi_{lk}^*(t)Pr(S_t = l|y^t, \vartheta)}{\sum_{j=1}^{K} \xi_{jkl}^*(t)Pr(S_t = j|y^t, \vartheta)}$$
References


