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Expectations formation under adaptive learning and evolutionary dynamics

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Abstract

Bounded rationality requires assumptions about ways in which rationality is constrained, and different assumptions are likely to lead to different economic predictions. In a simple forward looking model we compare adaptive learning and evolutionary dynamics as means to model the process of beliefs' adaptation in response to observed outcomes. We show that the two methods deliver different conclusions about equilibrium and transition dynamics, and we try to shed some light on the reasons for such discrepancies.

Key words: adaptive learning; evolutionary dynamics; expectations.

JEL classification: C62, D83, D84, E32.

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Expectations formation under adaptive learning and evolutionary dynamics

1 Introduction

Rational expectations (RE) represent a powerful way of closing and solving an economic model but they impose strong requirements on agents in terms of information and computational capabilities. In recent years, a growing number of studies have tried to understand the effect of replacing RE with more realistic ways of modelling agents' expectations. One popular method is adaptive learning, where agents are treated as econometricians that repeatedly adjust the parameter values in their model economy. Prominent examples are Sargent (1999), Evans and Honkapohja (2003), Evans, Honkapohja and Mitra (2009) and, for an extensive treatise on methodology and applications, Evans and Honkapohja (2001).

Other methods have been used to model the way agents revise their expectations over time that can be characterized as evolutionary methods. For example Sethi and Franke (1995) consider an environment with strategic complementarities where adaptive expectations compete with a costly sophisticated predictor under evolutionary pressures and find that the costly rational predictor survives asymptotically; Branch and McGough (2008) extend this analysis to a deterministic cobweb model, and find that similar results emerge.

The two approaches, adaptive learning and evolutionary dynamics, have also been studied together: for example Berardi (2011) combines evolutionary selection among heterogeneous classes of models with adaptive learning on the parameters of each model and finds that heterogeneous equilibria are possible when there is autocorrelation in the exogenous driving process for the economy; Branch and Evans (2006) propose the concept of misspecification equilibrium, where different underparameterized predictors are selected even in the limit under least-squares learning and dynamic predictor selection based on average profits; Guse (2010) considers a setting where agents can choose between a minimum state variable and a sunspot forecasting model and finds that with an ad hoc cost to using the sunspot predictor, heterogeneity cannot be sustained under the combined evolutionary-adaptive learning dynamics.

The relation between learning and evolution has been debated in biology for a long time: Sznajder, Sabelis and Egas (2012) recently reviewed some of the literature concerning the so called Baldwin effect (Baldwin, 1896), i.e., the hypothesis that adaptive learning, by improving fitness, accelerates evolution. Closer to our field of investigation, Marimon (1993) considers a game theoretical framework and compares key properties of adaptive learning (adaptation, experimentation and inertia) with their counterparts in evolutionary dynamics (reproduction, mutation and conservation).

We are not concerned here with the interplay between these two alternative schemes, but instead with the comparison of their properties as drivers of beliefs dynamics and selectors of equilibria. A key difference between adaptive learning and evolutionary schemes as ways of modelling the evolution of expectations is that in the first case the researcher usually endows agents with a single forecasting model and then allows them to optimize the parameters in such model through learning; in the second case instead the researcher selects a set of alternative forecasting models, and then allows agents to choose among this fixed set of rules based on their relative performance.[1] Anufriev and Hommes (2012) for example predetermine a set of rules, or heuristics, available to agents and investigate how the switching between these rules can explain results from experimental data.

Another fundamental point is that under adaptive learning schemes dynamics are driven by the difference between what is predicted by the forecasting model (beliefs) and actual outcomes, while under evolutionary schemes dynamics are driven by the difference in performance between different predictions (beliefs). If the performance is defined by the forecasting error, then, in the first case dynamics are driven by forecast errors in absolute terms, while in the second case by forecast errors in relative terms.

In order to investigate these issues, we chose a simple framework that is nevertheless rich enough to allow for an understanding of the key properties of these two schemes. In particular, we use a simple, one-dimensional forward looking model that allows for multiple equilibria. An essential feature of the setting proposed is the self-referentiality induced by expectations: without such self-referentiality, there would be only one forecasting model consistent with equilibrium, and that would always prevail in the end. In addition, we need a framework where the different schemes used to model belief dynamics (i.e., adaptive learning and evolutionary dynamics) could lead to the same outcome: this requires that the different heuristics used under replicator dynamics be nested into each other and within the perceived law of motion (PLM) used under adaptive learning. We achieve this by assuming only two heuristics (models for beliefs), one overparameterized with respect to the other (and with respect to the fundamental equilibrium). There are in principle two possibilities in terms of overparameterization: extra exogenous variables (sunspots), or extra lagged endogenous

[1] From an evolutionary perspective, only selection is allowed, but not mutation or crossover that could generate new alternatives.

variables. The first case has been analyzed in Berardi (20015) in a similar setting. We will instead compare here a model with the minimum possible number of state variables against one that is overparameterized with respect to it by the addition of a lagged endogenous variable: crucially, both models are consistent with RE equilibrium.

2 The model

We consider the simple, one-dimentional, forward looking model

$$y_t = A + BE_t y_{t+1} + v_t, \tag{1}$$

with v_t a zero mean, i.i.d. process with variance σ_v^2. $E_t y_{t+1}$ represents expectations held by agents, not necessarily rational.

This model has a minimum state variable (MSV) rational expectations (RE) equilibrium (see McCallum, 1982) given by

$$y_t = \frac{A}{1-B} + v_t \tag{2}$$

and a continuum of AR(1) RE equilibria in the form

$$y_t = -\frac{A}{B} + \frac{1}{B} y_{t-1} - \frac{1}{B} v_{t-1} + \varepsilon_t \tag{3}$$

where $\varepsilon_t = y_t - E_{t-1} y_t$. These equilibria are non-explosive for $|B| > 1$. If we set $\varepsilon_t \equiv v_t$, we obtain the MSV equilibrium with an $AR(1)$ representation, given appropriate initial conditions.

3 Adaptive learning

Adaptive learning is usually modelled through schemes such as recursive least squares (RLS) and stochastic gradient, but, more in general and for our purposes, could comprise any scheme where beliefs are measured against actual outcomes and updated in the direction of the error. We will assume here that agents use a RLS algorithm to form their expectations, and we will follow Evans and Honkapohja (2001) for the ensuing analysis of stability under learning (E-stability).

Consider first an homogenous setting. Assume agents use a PLM consistent with the MSV equilibrium

$$y_t = a, \tag{4}$$

where a is a constant to be recursively estimated from data. Then their expectations are

$$E_t y_{t+1} = a$$

and the ensuing actual law of motion (ALM) for the economy is

$$y_t = A + aB + v_t.$$

The so called T-map from PLM to ALM is given by

$$T(a) = A + aB$$

and the ensuing ODE representing learning dynamics[2] is given by

$$\dot{a} = A + aB - a$$

whose stability is governed by

$$\frac{\partial \dot{a}}{\partial a} = B - 1.$$

The MSV equilibrium is thus said to be E-stable if $B < 1$.

Consider now an economy where agents use a PLM consistent with the AR(1) equilibrium

$$y_t = a + b y_{t-1}. \tag{5}$$

Expectations are then given by

$$E_t y_{t+1} = a(1+b) + b^2 y_{t-1}$$

and the ensuing ALM is

$$y_t = [A + a(1+b)B] + Bb^2 y_{t-1} + v_t.$$

The ODEs are now

$$\begin{aligned}\dot{a} &= A + a(1+b)B - a \\ \dot{b} &= Bb^2 - b\end{aligned}$$

[2]See Evans and Honkapohja (2001) for details on the E-stability analysis.

and stability is governed by the Jacobian

$$J = \begin{bmatrix} (1+b)\,B-1 & aB \\ 0 & 2bB-1 \end{bmatrix}.$$

The eigenvalues of Jacobian J, evaluated at the AR(1) equilibrium values $(a,b) = \left(-\frac{A}{B}, \frac{1}{B}\right)$, are 1 and B, and thus such equilibrium is unstable under learning dynamics (E-unstable). The eigenvalues of Jacobian J evaluated at the MSV equilibrium $(a,b) = \left(\frac{A}{1-B}, 0\right)$ are instead -1 and $B-1$: for $B<1$ the system is locally asymptotically stable and the MSV equilibrium is therefore said to be strongly E-stable w.r.t. PLM (5). Agents can learn to discard the lagged endogenous variable and converge to the MSV solution.

3.1 Heterogeneous expectations

We consider now an economy where there are two groups of agents, one using (4) and the other using (5).

Group 1 is endowed with PLM_1, consistent with the MSV equilibrium

$$y_t = a_1$$

while group 2 is endowed PLM_2, consistent with the AR(1) equilibrium

$$y_t = a_2 + by_{t-1.}$$

Denoting by μ the fraction of agents using PLM_1, the ensuing aggregate expectations are

$$E_t y_{t+1} = \mu a_1 + (1-\mu)\left(a_2\,(1+b) + b^2 y_{t-1}\right)$$

and the ALM

$$y_t = A + B\left(\mu a_1 + (1-\mu)\,a_2\,(1+b)\right) + B\,(1-\mu)\,b^2 y_{t-1} + v_t.$$

The T-maps are

$$\begin{aligned} a_1 &= A + B\left(\mu a_1 + (1-\mu)\,a_2\,(1+b)\right) && (6)\\ a_2 &= A + B\left(\mu a_1 + (1-\mu)\,a_2\,(1+b)\right) && (7)\\ b_2 &= B\,(1-\mu)\,b^2. && (8) \end{aligned}$$

The stability of the ensuing system of ODEs is governed by the Jacobian

$$J = \begin{bmatrix} B\mu - 1 & B(1-\mu)(1+b) & B(1-\mu)a_2 \\ B\mu & B(1-\mu)(1+b) - 1 & B(1-\mu)a_2 \\ 0 & 0 & 2bB(1-\mu) - 1 \end{bmatrix}.$$

The two equilibria (fixed points of the system of ODEs) are the MSV, where $(a_1, a_2, b) = \left(\frac{A}{1-B}, \frac{A}{1-B}, 0\right)$, and the $AR(1)$, where $(a_1, a_2, b) = \left(-\frac{A}{B}, -\frac{A}{B}, \frac{1}{B(1-\mu)}\right)$.

We then evaluate the Jacobian at the two equilibria. At the MSV equilibrium we have

$$J_{MSV} = \begin{bmatrix} B\mu - 1 & B(1-\mu) & \frac{AB}{1-B}(1-\mu) \\ B\mu & B(1-\mu) - 1 & \frac{AB}{1-B}(1-\mu) \\ 0 & 0 & -1 \end{bmatrix},$$

which has eigenvalues $\{-1, -1, B-1\}$ and so the MSV equilibrium is E-stable for $B < 1$.

Instead at the AR(1) equilibrium we have

$$J_{AR(1)} = \begin{bmatrix} B\mu - 1 & B(1-\mu) + 1 & -A(1-\mu) \\ B\mu & B(1-\mu) & -A(1-\mu) \\ 0 & 0 & 1 \end{bmatrix},$$

whose eigenvalues are $\{-1, 1, B\}$ and so the equilibrium is E-unstable for any $\mu \in [0, 1]$.

Note that an heterogeneous solution with $(a_1, a_2, b) = \left(\frac{A}{1-B}, -\frac{A}{B}, \frac{1}{B(1-\mu)}\right)$ is not possible, as it is not a fixed point of the ODEs.

4 Evolutionary dynamics

Evolutionary dynamics are modelled here using replicator dynamics. Other schemes could be used, such as the Brock and Hommes (1997) dynamics based on a logit model, or even models of reinforcement learning. In general, any scheme where competing rules (beliefs/models) are measured in terms of performance (some "distance" from actual outcomes) and the best performing one(s) get reinforced.

A crucial issue in any evolutionary scheme without mutation/crossover is to decide the set of alternatives that will compete against each other. We will consider different cases, driven by considerations about the equilibria of the model. In each case we will have two models (PLMs) competing against each other.

We assume that there is a continuum of agents of unit mass, distributed on the unit interval, and the fraction of agents using each forecasting model evolves according to replicator dynamics driven by relative performance, as measured by mean squared errors (MSEs). In notation, the fraction of agents μ using PLM_1 evolves according to the ODE

$$\dot{\mu} = \mu (1 - \mu) \Delta \tag{9}$$

where

$$\Delta = MSE_2 - MSE_1, \tag{10}$$

with MSE_2 the mean squared error of forecasts made using PLM_2, and MSE_1 the mean squared error of forecasts made using PLM_1. Clearly equilibrium points of evolutionary dynamics are points where $\mu = 0, \mu = 1$ or $\Delta = 0$.

Local stability of equilibrium points under replicator dynamics is governed by $\frac{\delta \dot{\mu}}{\delta \mu}$. We have that

$$\frac{\delta \dot{\mu}}{\delta \mu} = (1 - 2\mu) \Delta + \mu (1 - \mu) \frac{\delta \Delta}{\delta \mu}.$$

In particular, at $\mu = 1$ or $\mu = 0$ local asymptotic stability is governed only by the sign of Δ: $\mu = 0$ is stable for $\Delta < 0$ while $\mu = 1$ is stable for $\Delta > 0$.

Using the mean squared error as a driver of evolutionary dynamics might seem to impose strong informational requirements on agents. An alternative would be to assume that agents use an adaptive process to estimate the MSE over time, such as

$$MSE_t = (1 - g_t) MSE_{t-1} + g_t (y_t - E_{t-1} y_t)^2$$

with g_t a small fixed or decreasing gain.[3] For simplicity and analytical tractability, we will assume in this work instead that agents have full knowledge of the relevant $MSEs$ at each point in time.

4.1 Competing models

There are two possible models available to agents at all times. The first, denoted, PLM_1, is consistent with the MSV equilibrium

$$y_t = a_1 \tag{11}$$

[3] In order to use this measure of performance in our analysis, we would also need to replace the continuous time replicator dynamics with a discrete time version.

while the second, PLM_2, is consistent with the AR(1) equilibrium

$$y_t = a_2 + by_{t-1}, \tag{12}$$

where parameters a_1, a_2 and b will be specified later for different scenarios.

With an endogenous fraction μ of agents using the first model and the remaining fraction $(1-\mu)$ using the second model, the ensuing aggregate expectations are

$$E_t y_{t+1} = \mu a_1 + (1-\mu)\left(a_2(1+b) + b^2 y_{t-1}\right)$$

and the ALM is given by

$$y_t = A + B(\mu a_1 + (1-\mu)a_2(1+b)) + B(1-\mu)b^2 y_{t-1} + v_t. \tag{13}$$

The fraction of agents using each model is endogenous and evolves according to the replicator dynamics system (9)-(10), where

$$\begin{aligned} MSE_1 &= E(y_t - a_1)^2 \\ MSE_2 &= E(y_t - (a_2 + by_{t-1}))^2 . \end{aligned}$$

Inserting the ALM into the equations for the MSEs we obtain

$$\begin{aligned} MSE_1 &= E\left(A + (B\mu - 1)a_1 + B(1-\mu)a_2(1+b) + B(1-\mu)b^2 y_{t-1} + v_t\right)^2 & (14)\\ MSE_2 &= E\left(A + B(\mu a_1 + a_2((1-\mu)(1+b)-1)) + \left(B(1-\mu)b^2 - b\right)y_{t-1} + v_t\right)^2 & (15) \end{aligned}$$

Fixing parameters (a_1, a_2, b) will allow us to derive conditions for the dynamics of μ.

4.2 PLMs consistent with RE equilibria

As seen above, there are two possible RE equilibria for the model (1). These equilibria define the two PLMs as follows: PLM_1 is given by

$$a_1 = \frac{A}{1-B}, \tag{16}$$

which is consistent with the MSV equilibrium; while PLM_2 is given by

$$(a_2, b) = \left(-\frac{A}{B}, \frac{1}{B}\right), \tag{17}$$

consistent with the AR(1)equilibrium. The ensuing aggregate expectations are given by

$$E_t y_{t+1} = \frac{A\left(B^2+\mu-1\right)}{B^2\left(1-B\right)} + \frac{1-\mu}{B^2} y_{t-1}$$

and the ALM is

$$y_t = \frac{A\left(B+\mu-1\right)}{B\left(1-B\right)} + \frac{1-\mu}{B} y_{t-1} + v_t. \tag{18}$$

Inserting the ALM into the equations for MSEs we obtain

$$\begin{aligned} MSE_1 &= E\left(\frac{A\left(\mu-1\right)}{B\left(1-B\right)} + \frac{1-\mu}{B} y_{t-1} + v_t\right)^2 \\ MSE_2 &= E\left(\frac{A\mu}{B\left(1-B\right)} - \frac{\mu}{B} y_{t-1} + v_t\right)^2. \end{aligned}$$

Using the fact that, from (18) and assuming stationarity of y_t,[4]

$$\begin{aligned} Ey_t &= \frac{A}{1-B} \\ Ey_t^2 &= \left(\frac{A}{1-B}\right)^2 + \frac{B^2}{B^2-\left(1-\mu\right)^2}\sigma_v^2 \end{aligned}$$

we have that

$$\begin{aligned} MSE_1 &= \frac{B^2\sigma_v^2}{B^2-\left(1-\mu\right)^2} \\ MSE_2 &= \frac{\left(B^2+2\mu-1\right)\sigma_v^2}{B^2-\left(1-\mu\right)^2}. \end{aligned}$$

Clearly $MSE_2 > MSE_1 \Leftrightarrow \mu < \frac{1}{2}$: the model used by the majority of agents delivers better forecasts and in the long run prevails under evolutionary dynamics. This means that both $\mu = 1$ and $\mu = 0$ are locally asymptotically stable equilibria, with separating point $\mu = \frac{1}{2}$, which is an unstable equilibrium. For $\mu > 1/2$, $\mu \to 1$, while for $\mu < 1/2$, $\mu \to 0$.

Proposition 1 *In an economy represented by (1), the AR(1) model (17) delivers better expectations (as measured by MSE) than the MSV model (16) if $\mu < \frac{1}{2}$: it follows that, starting from any $\mu < \frac{1}{2}$, the economy would converge to an equilibrium with $\mu = 0$. If $\mu > \frac{1}{2}$, instead, the MSV model dominates and the economy would converge to an equilibrium with $\mu = 1$.*

[4]Note, from ALM (18), that stationarity requires $|1-\mu| < |B|$. As $\mu \to 1$, this is satisfied $\forall B$, since nobody is using the PLM_2 anymore. As $\mu \to 0$, instead, the requirement is the usual one for the existence of stationary AR(1) solutions seen above, i.e., $|B| > 1$.

This result is particularly interesting if compared to the outcome under adaptive learning: in that case the AR(1) equilibrium was not E-stable so agents using an AR(1) model would not converge to equilibrium. Moreover, the MSV equilibrium was strongly E-stable w.r.t. the AR(1) model, which means that agents, starting in a neighborhood of the MSV equilibrium, would be able to learn to discard the additional lagged endogenous variable and the economy would converge to the MSV equilibrium. We can see instead that under evolutionary dynamics, contrary to learning, the economy can converge to an equilibrium where all agents use the overparameterized AR(1) model: it only requires that enough agents start using such model at the same time. This result is due to the fact that evolutionary dynamics are driven by the difference in performance between the two models, not the distance between the model forecasts and actual outcomes, as is the case under learning. Since actual outcomes depend on forecasts, if enough people make forecasts using the overparameterized PLM_2, forecasts made with such model have smaller MSE and can prevail under evolutionary dynamics.

4.3 Arbitrary beliefs on the lagged endogenous variable

We have seen that if enough agents use an AR(1) model consistent with the AR(1) RE equilibrium, under replicator dynamics the economy can converge to such equilibrium. We will show here an even stronger result, i.e., that the AR(1) model does not need to be consistent with the AR(1) RE equilibrium for it to prevail.

The question we consider in this section can be reformulated in this way: suppose we are in the MSV equilibrium, where all agents use PLM_1 with $a_1 = \frac{A}{1-B}$, and then at some point some agents introduce a lagged variable with arbitrary coefficient in their forecasting model, therefore using PLM_2 summarized by $(a_2, b) = \left(\frac{A}{1-B}, b\right)$, with arbitrary b. Could these agents take over in the economy? In other words, could we move from a situation where all agents use the MSV model to one where all agents use an AR(1) model with arbitrary autoregressive coefficient?

We know that, under adaptive learning, agents would learn to discard the additional lagged endogenous variable, provided they start in a neighborhood of the fundamental equilibrium, since we have seen that the MSV solution is strongly E-stable w.r.t. such overparameterization. What would happen instead under evolutionary dynamics?

Starting from equation (13) with $a_1 = a_2 = A/(1-B)$ and arbitrary b we obtain the

ALM

$$y_t = \frac{A + AB\left(1-\mu\right)b}{1-B} + B\left(1-\mu\right)b^2 y_{t-1} + v_t \tag{19}$$

which implies, assuming stationarity of y_t,

$$\begin{aligned} Ey_t &= \frac{A\left(1 + B\left(1-\mu\right)b\right)}{\left(1-B\right)\left(1-B\left(1-\mu\right)b^2\right)} \\ Ey_t^2 &= \left[\frac{A\left(1 + B\left(1-\mu\right)b\right)}{\left(1-B\right)\left(1-B\left(1-\mu\right)b^2\right)}\right]^2 + \frac{\sigma_v^2}{1-\left(B\left(1-\mu\right)b^2\right)^2}. \end{aligned}$$

We can then compute

$$\begin{aligned} MSE_1 &= E\left(y_t - \frac{A}{1-B}\right)^2 = \frac{b^2\left(1+b\right)^2 A^2 B^2 \left(1-\mu\right)^2}{\left(1-B\right)^2\left(1-B\left(1-\mu\right)b^2\right)^2} + \frac{1}{\left(1-B\left(1-\mu\right)b^2\right)^2}\sigma_v^2 \\ MSE_2 &= E\left(y_t - \frac{A}{1-B} - by_{t-1}\right)^2 = \frac{b^2 A^2\left(1-B\left(1-\mu\right)\right)^2}{\left(1-B\right)^2\left(1-B\left(1-\mu\right)b^2\right)^2} + \frac{b^2\left(1-2B\left(1-\mu\right)b\right)+1}{\left(1-B\left(1-\mu\right)b^2\right)^2}\sigma_v^2 \end{aligned}$$

and

$$\Delta = \frac{A^2 b^2\left[1-2B\left(1-\mu\right) + b\left(-2B^2\left(1-\mu\right)^2\right) + b^2\left(-B^2\left(1-\mu\right)^2\right)\right]}{\left(1-B\right)^2\left(1-B\left(1-\mu\right)b^2\right)^2} + \frac{b^2\left(1-2B\left(1-\mu\right)b\right)}{\left(1-B\left(1-\mu\right)b^2\right)^2}\sigma_v^2.$$

Consider first the case without noise, i.e., $\sigma_v^2 \equiv 0$. Clearly then the sign of Δ depends on the quadratic (in b) polynomial

$$\tilde{\Delta} = 1 - 2B\left(1-\mu\right) + b\left(-2B^2\left(1-\mu\right)^2\right) + b^2\left(-B^2\left(1-\mu\right)^2\right) \tag{20}$$

whose roots are

$$\begin{aligned} b_1 &= -\frac{1}{B\left(1-\mu\right)}, \\ b_2 &= -2 + \frac{1}{B\left(1-\mu\right)}. \end{aligned}$$

Note that, besides the two homogeneous equilibria $\mu = 0$ and $\mu = 1$, there is the possibility of an heterogeneous equilibrium here, when $\tilde{\Delta} = 0$, i.e., when

$$\mu = 1 + \frac{1}{Bb} \text{ or } \mu = 1 - \frac{1}{B\left(2+b\right)}.$$

4.3.1 Homogeneous equilibria

We consider first the issue of stability for the homogeneous equilibria. Defining $b_i < b_j$, we have that for $b_i < b < b_j$, $\Delta > 0$ while for $b < b_i \cup b > b_j$, $\Delta < 0$. We need to distinguish the cases for positive and negative B.

If $B < 0$, then $b_1 > b_2 \ \forall \ \mu \in [0, 1]$. Moreover, since

$$\begin{aligned} \lim_{\mu \to 1} b_1 &= +\infty; \ \lim_{\mu \to 1} b_2 = -\infty \\ \lim_{\mu \to 0} b_1 &= -\frac{1}{B} > 0; \ \lim_{\mu \to 0} b_2 = \frac{1}{B} - 2 < 0 \end{aligned}$$

it follows that:

- at $\mu = 1$, $\forall b \in (-\infty, +\infty)$, $\Delta > 0$, and therefore $\mu = 1$ is locally asymptotically stable;
- at $\mu = 0$, $\Delta < 0$ iff $b < \frac{1}{B} - 2$ or $b > -\frac{1}{B}$: in such cases $\mu = 0$ is locally asymptotically stable.

If instead $B > 0$, since

$$\begin{aligned} \lim_{\mu \to 1} b_1 &= -\infty; \ \lim_{\mu \to 1} b_2 = \infty \\ \lim_{\mu \to 0} b_1 &= -\frac{1}{B}; \ \lim_{\mu \to 0} b_2 = -2 + \frac{1}{B} \end{aligned}$$

it follows that:

- at $\mu = 1$, $\forall b \in (-\infty, +\infty)$, $\Delta > 0$, and therefore $\mu = 1$ is locally asymptotically stable;
- at $\mu = 0$, we need to distinguish between two cases:
  - if $0 < B < 1$, then $b_2 > b_1$: for $b < b_1$ or $b > b_2$, $\Delta < 0$ and therefore $\mu = 0$ is locally asymptotically stable;
  - if $B > 1$, then $b_1 > b_2$: for $b < b_2$ or $b > b_1$, $\Delta < 0$, and therefore $\mu = 0$ is locally asymptotically stable.

We summarize results in the following two propositions:

Proposition 2 *Case with $A > 0$, $\sigma_v^2 = 0$. The homogeneous equilibrium $\mu = 1$ is locally asymptotically stable under replicator dynamics (since, at $\mu = 1$, $\Delta > 0$, $\forall b$).*

Proposition 3 *Case with $A>0$, $\sigma_v^2=0$. There exist values of b for which the homogeneous equilibrium $\mu=0$ is locally asymptotically stable under replicator dynamics (i.e., they imply $\Delta<0$ at $\mu=0$). Such values depend on B and are:*

$$\begin{aligned} b &< -\frac{1}{B} \cup b > -2+\frac{1}{B}, \textit{ for } 0<B<1 \\ b &< -2+\frac{1}{B} \cup b > -\frac{1}{B} \textit{ for } B<0 \textit{ or } B>1. \end{aligned}$$

Note that, with $a=A/(1-B)$, there is no "right" value of b, since $b=\frac{1}{B(1-\mu)}$ is correct only in the $AR(1)$ equilibrium with $a=-A/B$. In the MSV equilibrium, with $a=A/(1-B)$, b should be 0.

We can see that the relative performance of the two models depends on the structure of the economy (B) and the fraction of agents using each model (μ). As more and more agents use the MSV model ($\mu\rightarrow 1$), this model dominates for any value chosen for b. As μ decreases to 0, though, the range of values for b that allow the MSV model to prevail shrinks, but it still remains non negligible even as $\mu\rightarrow 0$.

Note that for $B=0$ (no self-referentiality in the structural model), the MSV solution always prevails: it is the self-referentiality that allows evolutionary dynamics to introduce additional variables into the equilibrium, since the fitness measure (here the MSE) becomes endogenous to the choice of agents.

Propositions (2) an (3) imply that both $\mu=1$ and $\mu=0$ can be locally asymptotically stable equilibria for some parameterizations: it follows that in such cases there need to be a threshold $\tilde{\mu}$ that separates the basins of attractions of these equilibria, i.e., for any given b, there must be a $\tilde{\mu}$ such that $\forall\mu>\tilde{\mu}$, $\Delta>0$ (and $\mu\rightarrow 1$), and $\forall\mu<\tilde{\mu}$, $\Delta<0$ (and $\mu\rightarrow 0$). This threshold is defined as the value of μ such that $\tilde{\Delta}=0$ and such value depends on the arbitrary b. This is the heterogeneous equilibrium discussed in the next subsection.

4.3.2 Heterogeneous equilibrium

The condition for the existence of an heterogeneous equilibrium under evolutionary dynamics, where both $PLMs$ are used simultaneously by some agents, is

$$\tilde{\Delta}=1-2B(1-\mu)+b\left(-2B^2(1-\mu)^2\right)+b^2\left(-B^2(1-\mu)^2\right)=0$$

or equivalently

$$\left(1-2B-2B^2b-B^2b^2\right)+\mu\left(2B+4B^2b+2B^2b^2\right)+\mu^2\left(-2B^2b-B^2b^2\right)=0,$$

which is quadratic in μ. Only one root can be $\in (0,1)$ when conditions on Proposition (3) are satisfied (there need to be an odd number of roots for Propositions (2) and (3) to hold simultaneously). The two roots are

$$\begin{aligned}\mu_1 &= 1+\frac{1}{Bb}\\ \mu_2 &= 1-\frac{1}{B(2+b)}.\end{aligned}$$

The threshold dividing the two basins of attraction for the equilibria $\mu = 0$ and $\mu = 1$ is therefore $\tilde{\mu} = \mu_i \in (0,1)$, which is an unstable resting point for the evolutionary dynamics (i.e., $\frac{\delta \dot{\mu}}{\delta \mu} > 0$ at $\tilde{\mu}$).

This means that, unless the economy starts out with exactly $\tilde{\mu}$ agents using PLM_1, dynamics will converge to one of the two homogeneous equilibria, depending on the specific parameterization of the model and the initial condition for μ.

Note that the threshold separating the two stable equilibria depends on B and b. For example, with $B > 0$ and $b > 0$, the relevant root is μ_2. Clearly, μ_2 is larger the larger are B and b: as the feedback from expectations to outcomes increases, or as the correlation parameter in beliefs increases, fewer agents with the AR(1) model are needed in order to have $\tilde{\Delta} < 0$ and the AR(1) model to be superior.

4.3.3 General case with noise

Now we look at the general case with noise, i.e., $\sigma_v^2 > 0$. Remember that

$$\Delta = \frac{A^2b^2\left[1-2B(1-\mu)+b\left(-2B^2(1-\mu)^2\right)+b^2\left(-B^2(1-\mu)^2\right)\right]}{(1-B)^2\left(1-B(1-\mu)b^2\right)^2}+\frac{b^2(1-2B(1-\mu)b)}{\left(1-B(1-\mu)b^2\right)^2}\sigma_v^2.$$

The sign of Δ depends on

$$\hat{\Delta} = A^2b^2\left[1-2B(1-\mu)+b\left(-2B^2(1-\mu)^2\right)+b^2\left(-B^2(1-\mu)^2\right)\right]+b^2(1-2B(1-\mu)b)(1-B)^2\sigma_v^2.$$

Let's consider first the case with $A \equiv 0$. Then

$$\hat{\Delta} = b^2(1-2B(1-\mu)b)(1-B)^2\sigma_v^2$$

and $\hat{\Delta} > 0 \Leftrightarrow (1-2B(1-\mu)b) > 0$, or $B(1-\mu)b < \frac{1}{2}$. This inequality has a simple interpretation: it is the condition such that the quadratic distance between expected and actual impact of the lagged endogenous variable is smaller for agents using PLM_1 than for agents using PLM_1. In fact, the maps from PLMs to ALM for the coefficient on the lagged endogenous variable are

$$\begin{aligned} PLM_1 &: \quad 0 \rightarrow B\left(1-\mu\right)b^2 \\ PLM_2 &: \quad b \rightarrow B\left(1-\mu\right)b^2. \end{aligned}$$

The difference in squared deviations is

$$\left(B\left(1-\mu\right)b^2 - b\right)^2 - \left(B\left(1-\mu\right)b^2 - 0\right)^2$$

or

$$b^2\left(1 - 2B\left(1-\mu\right)b\right)$$

whose sign condition reduces to the condition seen above for the sign of $\hat{\Delta}$. In particular, $\mu = 1$ is always a stable equilibrium, while $\mu = 0$ is a stable equilibrium if

$$Bb > \frac{1}{2}. \tag{21}$$

Moreover, for a given pair (B, b) that satisfies restriction (21), the threshold $\tilde{\mu}$ separating the two basins of attraction for the equilibria $\mu = 0$ and $\mu = 1$ is defined as

$$\tilde{\mu} = 1 - \frac{1}{2Bb}$$

which is an unstable equilibrium of the replicator dynamics.

Proposition 4 *Case with $A = 0$, $\sigma_v^2 > 0$. The equilibrium $\mu = 1$ is always locally asymptotically stable under replicator dynamics.*

Proposition 5 *Case with $A = 0$, $\sigma_v^2 > 0$. The equilibrium $\mu = 0$ is locally asymptotically stable under replicator dynamics if $Bb > \frac{1}{2}$. Moreover, starting from any $\mu < 1 - \frac{1}{2Bb}$, the economy will converge to the equilibrium $\mu = 0$ under replicator dynamics if $Bb > \frac{1}{2}$.*

With $A \neq 0$, the constant term and the AR(1) term interact in the determination of the relative performance of the two models and can compensate for each other, so the condition for the sign restriction of Δ is less straightforward to understand:

$$\hat{\Delta} > 0 \Leftrightarrow \frac{A^2}{\left(1-B\right)^2}\left(1 - B\left(1-\mu\right)\left(2 + B\left(1-\mu\right)b\left(2+b\right)\right)\right) > b^2\left(2B\left(1-\mu\right)b - 1\right)\sigma_v^2. \tag{22}$$

The r.h.s. of the inequality is the term related to the AR(1) component, affected by the size of the noise in the system, and its contribution to $\hat{\Delta}$ is positive if $B\left(1-\mu\right)b < \frac{1}{2}$, which is the condition seen above for $A = 0$. The term on the l.h.s. is due to the constant in the model, but it interacts with the AR(1) component: its contribution to $\hat{\Delta}$ is positive if $1 > B(1-\mu)(2 + B(1-\mu) b(2+b))$.

From condition (22), it follows that $\mu = 0$ is locally asymptotically stable if

$$\frac{A^2}{(1-B)^2}(1 - B(2 + Bb(2+b))) < b^2 (2Bb - 1)\sigma_v^2, \tag{23}$$

while $\mu = 1$ is always locally asymptotically stable.

Proposition 6 *Case with $A > 0$, $\sigma_v^2 > 0$. The equilibrium $\mu = 1$ is always locally asymptotically stable under replicator dynamics.*

Proposition 7 *Case with $A > 0$, $\sigma_v^2 > 0$. The equilibrium $\mu = 0$ is locally asymptotically stable under replicator dynamics if condition (23) is satisfied.*

We can see that as $A \to 0$, condition (23) reduces to condition (21). As $\sigma_v^2 \to 0$ instead, (23) reduces to conditions presented in Proposition (3)

4.4 A simpler case

To help gain insight into the previous results, we consider here an even simpler case, which will allow us to highlight some of the issues discussed above.

Consider the case with $A = 0$, $v_t \equiv 0$ and the two PLMs given by

$$\begin{aligned} PLM_1 &: y_t \equiv 0 \\ PLM_2 &: y_t = a, \end{aligned}$$

where a is to be estimated under learning, and fixed under evolutionary dynamics.

4.4.1 Adaptive learning

Assume that only a fraction $(1-\mu)$ of agents implements adaptive learning, while the remaining fraction μ use PLM_1, and therefore believe y_t to always be at its steady state level. Aggregate expectations are $(1-\mu)a$ and the ALM is

$$y_t = B(1-\mu)a. \tag{24}$$

Learning analysis leads to the ODE

$$\dot{a} = (B(1-\mu) - 1)a$$

and stability of the fundamental equilibrium with $a = 0$ requires

$$B(1-\mu) < 1. \tag{25}$$

4.4.2 Evolutionary dynamics

Under evolutionary dynamics, beliefs parameter a is fixed and what evolves over time is instead μ, the fraction of agents using each PLM. For any arbitrary a, aggregate expectations are $(1-\mu)a$ and the ALM is again (24). The fraction μ evolves according to replicator dynamics

$$\dot{\mu} = \mu(1-\mu)\Delta$$

where

$$\Delta = MSE_2 - MSE_1.$$

Mean squared errors are given by

$$\begin{aligned} MSE_1 &= E(B(1-\mu)a - 0)^2 \\ MSE_2 &= E(B(1-\mu)a - a)^2 \end{aligned}$$

and therefore

$$\Delta = a^2(1 - 2B(1-\mu)).$$

Dynamics depend on the sign of Δ and clearly

$$\Delta > 0 \Leftrightarrow B(1-\mu) < \frac{1}{2}. \tag{26}$$

Clearly conditions (25) and (26) differ from each other. The reason of this difference is that dynamics of a under adaptive learning are driven by the distance between PLM_2 and ALM, while dynamics of μ under replicator dynamics are driven by the difference in the distance between each PLM and the ALM, as only relative performance matters. Evolutionary dynamics therefore impose a less restrictive condition for non-fundamental beliefs to prevail: forecast errors made using the overparameterized model don't need to vanish over time (as it would be required under learning), they just need to remain smaller than the errors made using the alternative model.

5 Two examples

We present here two examples that show the relevance of the previous results for our understanding of what long run outcomes can emerge in macroeconomic models under alternative expectations formation mechanisms. We consider a model with positive feedback from expectations to outcomes, and one with negative feedback. In both cases, the long run outcome under adaptive learning and evolutionary dynamics differ, with adaptive learning always favouring the MSV equilibrium, which implies that, respectively, inflation and consumption are white noise variables, while evolutionary dynamics allow for the emergence of persistency in these processes.

5.1 Positive feedback

Consider the simple Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + v_t \tag{27}$$

which is a simplification of the standard NK Phillips curve where output is assumed to be fixed at the equilibrium level, so inflationary pressures can come only out of expectations and exogenous cost push shocks.

The fundamental, or MSV, RE equilibrium is represented by

$$\pi_t = v_t. \tag{28}$$

Under adaptive learning, for the usual range of values $0 < \beta < 1$, the MSV equilibrium is E-stable, while the $AR(1)$ is not: the economy will therefore converge to the fundamental equilibrium over time.

Under evolutionary dynamics, instead, we have that, with $0 < \beta < 1$, $\mu = 0$ can be an equilibrium provided that b, which defines beliefs in PLM_2

$$\pi_t = b\pi_{t-1} \tag{29}$$

is such that $\beta b > 1/2$. In this case, inflation will then follow the process (ALM)

$$\pi_t = \beta b^2 \pi_{t-1} + v_t. \tag{30}$$

For example, with $\beta = .97$, any $b > 0.515$ would guarantee this outcome. Adding the restriction of stationarity for inflation, i.e., $|\beta b^2| < 1$,[5] it means that the range of values for which $\mu = 0$ is an equilibrium is $.515 < b < 1.015$. Moreover, starting from any $\mu < \tilde{\mu} = 1 - \frac{1}{2Bb}$, the economy would converge to the equilibrium $\mu = 0$. For example, with $\beta = .97$ and $b = .99$, such threshold is equal to 0.4793: just less than half the population needs to start use the AR(1) model for it to prevail.

5.2 Negative feedback

We consider now the cash in advance model analyzed in Evans, Honkapohja and Marimon (2007). They show that the dynamics of that economy can be summarized by a univariate reduced form model that represents an optimality condition for consumption, and in the linearized version takes the form

$$c_t = \alpha E_t c_{t+1} + g_t,$$

where c_t is consumption of the cash good (i.e., a good that has to be paid in cash, and is therefore subject to the cash in advance constraint) and g_t is (net) government spending, here taken to be a random variable with zero mean and constant variance for simplicity. Parameter α depends, among other things, on the coefficient of risk aversion σ (which is the inverse of the intertemporal elasticity of substitution $1/\sigma$). The model implies that $\sigma > 1 \Leftrightarrow \alpha < 0$: an expected increase in consumption tomorrow decreases consumption today.

The MSV equilibrium takes the form of a "noisy steady state" and is given by

$$c = g_t,$$

which is stable under adaptive learning for any value of $\alpha < 0$ (the stability requirement is $\alpha < 1$). On the contrary, the $AR(1)$ solution is unstable under learning.[6]

Turning now to outcomes under evolutionary dynamics, the condition for $\mu = 0$ to be an equilibrium is $\alpha b > 1/2$, where b defines beliefs in PLM_2

$$c_t = bc_{t-1}.$$

[5] Note that, with $\beta > 0$, in order to be possible to find a b such that $\beta b > \frac{1}{2}$ and $\left|\beta b^2\right| < 1$,the model must satisfy the restriction $\beta > \frac{1}{4}$.

[6] Evans, Honkapohja and Marimon (2007) show that when $\sigma > 1$, there exist Markov stationary sunspot equilibria that are stable under learning.

Given that $\alpha < 0$, as long as $b < \frac{1}{2\alpha}$the economy will then converge to the ALM

$$c_t = \alpha b^2 c_{t-1} + g_t.$$

With the additional requirement that $|\alpha b^2| < 1$ for stationarity of consumption,[7] we have that, for example with $\alpha = -1$, an equilibrium with $\mu = 0$ would emerge if agents were to use a PLM with $-1 < b < -\frac{1}{2}$. While the MSV solution implies white noise consumption, and such solution would be selected by adaptive learning, under evolutionary dynamics we would instead have an AR(1) process with negative correlation for consumption.

6 Discussion

We have proposed a model where evolutionary dynamics and adaptive learning deliver different results in terms of selection of beliefs and equilibrium outcomes: while adaptive learning always favours the fundamental (MSV) model, for some parameterizations evolutionary dynamics allow beliefs based on an overparameterized model to prevail. The key issue in understanding this result is the self-referentiality of (1), since the environment changes with the behavior of agents. This means that the measure of fitness for beliefs, their forecasting accuracy, is endogenous to the choice of agents, and thus multiple equilibria can emerge. Consider in fact the alternative model with $B = 0$: it follows from Proposition 5 that in this case $\mu = 0$ cannot be an equilibrium and beliefs based on the MSV model would always dominate under evolutionary dynamics.[8]

Self-referentiality has a stronger impact on evolutionary dynamics than on adaptive learning because evolutionary dynamics are based on the relative performance of competing beliefs, rather than the absolute performance of beliefs against actual outcomes as it is the case under adaptive learning. In a sense, therefore, beliefs under evolutionary dynamics are less constrained by outcomes, as they only need to prevail in relative terms.

A similar result was derived by Berardi (2015) for the case of sunspot equilibria: while equilibria based on sunspots are learnable only under specific parameterizations, beliefs based on a sunspot always prevail under replicator dynamics (even though in that case, because of a time-varying resonant condition on the sunspot, sunspot equilibria could not emerge

[7] Note that, with $\alpha < 0$, in order to be possible to find a b such that $\alpha b > \frac{1}{2}$ and $\left|\alpha b^2\right| < 1$, the model must satisfy the restriction $\alpha < -\frac{1}{4}$.

[8] Of course, it wouldn't matter for aggregate dynamics which model agents use for their forecasts, since these forecasts do not impact on actual outcomes. Still, forecasts quality could matter at the individual level.

endogenously under replicator dynamics).

Another way to look at it is that adaptive learning takes place at the individual level, while evolution takes place at population level. Under learning, at individual level, expected outcomes are measured against realized ones, and if unsatisfactory, current actions are changed. Under evolutionary dynamics, instead, strategies are played against each other in a population, and the better ones are reinforced.

The relation between learning and evolutionary dynamics has been studied extensively in the literature, especially in game theory. Fudenberg and Levine (1998), ch. 3, derive replicator dynamics from models of individual learning. Borgers and Sarin (1997) also show, in a game theoretical framework, that in a continuous time limit a learning model based on reinforcement of strategies with higher payoff converges to the replicator dynamics of evolutionary game theory. The key to these results is that in those cases, even under learning, beliefs (or, in their case, strategies) are played against each other, and not against actual outcomes.

Our analysis is based on a measure of performance summarized by the MSE: other measures could be conceived of, and it would be interesting to see whether alternative choices change results. In particular, it would be interesting to study the case where evolutionary dynamics are based on some measure of short run performance (such as the forecast error in the previous n periods).

7 Conclusions

In this work we compare adaptive learning and evolutionary dynamics as means to model the evolution of beliefs. Using a model that allows for multiple equilibria, we investigate what equilibrium outcomes can be expected to emerge under different belief dynamics. While specific conditions somewhat differ from case to case for the different specifications considered, in general we find that while adaptive learning favours the fundamental, or minimum state variables equilibrium, evolutionary dynamics can instead select beliefs based on an overparameterized model and therefore lead to non-fundamental solutions. This result is particularly important since it is not clear what is the best way to model belief dynamics in macroeconomic models, and we have shown here that different choices determine the set of outcomes that one can expect to emerge in the economy.

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