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On the fragility of sunspot equilibria under learning and evolutionary dynamics

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Abstract

In this paper we investigate the possibility of sunspot equilibria to emerge from a process of learning and adaptation on agents' beliefs. To such end, we consider both finite state Markov sunspots and sunspots in autoregressive form and derive conditions for the existence of an heterogeneous equilibrium where only a fraction of agents condition their forecasts on the sunspot: such conditions impose restrictions across primitive parameters, which are the equivalent, in a heterogeneous setting, of the resonant conditions found in the literature for homogeneous equilibria. We then show that evolutionary dynamics on predictor selection imply that such restrictions need to evolve endogenously with population shares, and argue that such requirement questions the possibility of sunspot equilibria to emerge through a process of evolution and adaptation on agents' beliefs. It follows that, in order for a sunspot equilibrium to obtain, all agents must simultaneously coordinate on using the same sunspot variable at the same time.

Key words: Sunspots; heterogeneity; expectations; learning; evolutionary dynamics.
JEL classification: C62, D83, D84, E32.

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On the fragility of sunspot equilibria under learning and evolutionary dynamics

1 Introduction

Sunspot equilibria are an intriguing possibility, as they open the door to fluctuations in economic activity driven purely by agents' expectations and disconnected from economic fundamentals. From the seminal works of Azariadis (1981) and Cass and Shell (1983), the possibility of self-fulfilling equilibria is well known among economists: because agents expect some particular state of the system to get realized in the future, that very state emerges as an equilibrium outcome for the economy.

While the early works considered the possibility of finite state Markov sunspot equilibria (and in particular 2 state sunspot equilibria - 2-SSE), in the business cycles literature a different class of sunspot solutions is more frequently considered, with an autoregressive-moving average (ARMA) form. Examples are found in McCallum (1983) and Farmer (1993).

As Evans and McGough (2011) recently remarked, the fact that sunspot equilibria are theoretically possible in a model does not make them necessarily relevant from an economic perspective, as it might not be possible for agents to coordinate on such equilibria. Because of this, a number of authors have tried to understand the conditions under which sunspot equilibria are learnable. Woodford (1990), Evans and Honkapohja (1994a) and Evans and Honkapohja (2003) analyze learnability for finite state Markov sunspot equilibria, while Evans and Honkapohja (1994b) and Evans and McGough (2005a, 2005b) show that also sunspot solutions in ARMA form can be learnable. In particular, Evans and McGough (2005a) demonstrate how in this last case the representation of the solution is crucial for its learnability properties. Building on this result, Evans and McGough (2011) show in a purely forward looking model that when finite state Markov sunspots equilibria are stable under learning, all sunspot equilibria are, provided a common factor representation is used.

All these works take a representative agent approach, and consider only the possibility of all agents conditioning their expectations on an extraneous sunspot component. Heterogeneity in expectations, though, has attracted increasing interest in the recent literature, as it is recognized that it represents a real world feature that economists must take into account in their understanding of expectations formation. In particular, the possibility of different predictors being endogenously chosen on the basis of their relative performance has been investigated in different contexts. From the seminal work of Brock and Hommes (1997), a number of works have analyzed the evolutionary selection of forecasting rules and their impact on economic outcomes. Recent examples include Branch and Evans (2006), Hommes (2009), Guse (2010) and Berardi (2011).

Much less investigated so far has been the link between heterogeneity and sunspot equilibria. A notable exception is Berardi (2009), who shows the possibility of heterogeneous equilibria, where only a fraction of agents use a sunspot variable in their forecasts, to emerge in a purely forward looking model, but who also points out the fragility of such equilibria under predictor choice dynamics. If agents are allowed to choose endogenously whether to include or not a sunspot in their forecasting model, based on a mean squared error measure of performance, it does not exist an equilibrium where only a fraction of agents uses the sunspot.

The aim of this paper is to study the conditions that are required for a sunspot equilibrium to emerge from a process of learning and adaptation on agents' beliefs. The main contribution will be to provide general results about the fragility of sunspot equilibria when learning and evolutionary dynamics are taken into account. In particular, we will show that evolutionary dynamics are intrinsically incompatible with sunspot equilibria, as they require the (exogenous) sunspot's statistical properties to endogenously change together with population dynamics.

Friedman (1991) advocates for evolutionary dynamics as models of repeated anonymous strategic interaction, where actions that are more "fit", given the distribution of behaviors, tend over time to displace actions that lead to lower rewards. In our setting, the "game" is between agents conditioning their forecasts only on fundamentals, and agents using also a sunspot variable in their model. Would either group prevail in the long run? Note that those agents using only fundamentals have a model that is in fact mispecified (or underparameterized), since the sunspot, through the expectations of the other group, enters into the dynamics of the economy. It will turn out that the answer to our question will not depend on the stability of the evolutionary process but will be instead of a more general nature: evolutionary dynamics kill the possibility of sunspot equilibria altogether, by imposing a time-varying restriction between population dynamics and sunspots.

We will present our argument both for finite state Markov sunspots and for sunspots in autoregressive form and show that the results are not a feature of the particular form assumed for the sunspot, but depend instead on the restrictions that evolutionary dynamics impose on it.

We will model agents' endogenous selection of forecasting rules by using replicator dynamics, which represents the evolution of the fraction of agents using each of the possible predictors available. The concept of replicator dynamics is popular in game theory and it is used to model evolutionary dynamics of strategies in the population of players. While it is borrowed from biology, where it was first introduced by Taylor and Jonker (1978) to formalize the notion of evolutionarily stable strategy, Borgers and Sarin (1997) give it a learning interpretation at the individual level. Fudenberg and Levine (1998) provide an extensive treatment in game theory, while Sethi and Franke (1995), Branch and McGough (2008) and Guse (2010) have applied it to macroeconomic settings. We will then show in a later section that our results do not hinge on the specific choice of model for evolutionary dynamics, and present our argument using the Brock and Hommes (1997) model of evolution in predictor choices.

In order to pin down parameters in each forecasting model, or perceived law of motion (PLM), we will follow a growing literature in macroeconomics and assume that agents act as econometrician and recurrently estimate those parameters using techniques such as recursive least-squares (for a detailed treatment of the concepts and techniques used in this literature, see Evans and Honkapohja, 2001). In equilibrium, parameter values in agents' forecasting models will therefore minimize the mean squared error for that specific class of models.

The plan of the paper is as follows: in Section 2 we present finite state Markov sunspot equilibria and extend them to a heterogeneous setting; in Section 3 we present equilibria with sunspots in AR(1) form and extend them to a heterogeneous setting; in Section 4 we introduce evolutionary dynamics, show how heterogeneity impacts on the resonant condition for the existence of sunspot equilibria and derive implications for the possibility of sunspot equilibria to emerge endogenously in an economy; Section 5 considers alternative predictor choice dynamics; Section 6 concludes.

2 Finite state Markov sunspot equilibria

The early literature on sunspot equilibria focused on finite state Markov equilibria, see for example Azariadis (1981), Azariadis and Guesnerie (1986), Guesnerie (1986) and Chiappori and Guesnerie (1989).

To fix ideas, consider the univariate linear model

$$y_t = \beta E_t^* y_{t+1}, \tag{1}$$

where $E_t^* y_{t+1}$ represents expectations held by agents, not necessarily rational.

Under rational expectations (RE), the fundamental equilibrium takes the form

$$y_t = 0,$$

but this is not the only possible equilibrium under RE. A finite state Markov sunspot equilibrium would take the form

$$y_t = \bar{y}_i \text{ when } s_t = s_i \text{ for } i = 1, ..., k, \tag{2}$$

where the sunspot $s_t \in \{s_1, ..., s_k\}$ is an exogenous, stationary finite state Markov process with fixed transition probabilities $\pi_{ij} = P(s_{t+1} = s_j | s_t = s_i)$. Solutions of the form (2) are called finite Markov stationary sunspot equilibria (SSEs), or k-state sunspot equilibria (k-SSEs). Considerable attention has been given in particular to the special case 2-SSEs, on which we will focus here. Note though that the argument of this paper carries out to any k-SSE.

For (2), with $k = 2$, to be a RE solution to (1) we need

$$\begin{aligned} \bar{y}_1 &= \beta \left(\pi_{11} \bar{y}_1 + \pi_{12} \bar{y}_2 \right) \\ \bar{y}_2 &= \beta \left(\pi_{21} \bar{y}_1 + \pi_{22} \bar{y}_2 \right) \end{aligned}$$

or

$$\bar{y} = \beta \Pi \bar{y}, \tag{3}$$

where Π is the matrix of transition probabilities and $\bar{y} = [\bar{y}_1, \bar{y}_2]'$. It is well known (see, e.g., Evans and Honkapohja (2003a)) that conditions for the sunspot solutions to exist are

$$\pi_{11} + \pi_{22} - 1 = \beta^{-1} \tag{4}$$
$$\left(1 - \pi_{22}\right) \bar{y}_1 + \left(1 - \pi_{11}\right) \bar{y}_2 = 0, \tag{5}$$

where condition (4) is usually referred to as a "resonant frequency condition" (see Evans and Honkapohja, 2003b) since it imposes a restriction on the transition probabilities of the Markov sunspot variable.

It is also well known (see Evans and Honkapohja (2003b), Evans and McGough (2011)) that the stationary sunspot equilibrium (3) is stable under learning if $\beta < -1$. This means that if all agents use a model consistent with (2), they would be able to learn the sunspot equilibrium provided such condition holds.

2.1 Heterogeneous solutions

We introduce now heterogeneity and consider the possibility of solutions where only a fraction of agents uses the sunspot. We will see that this requirement modifies the resonant condition for the existence of an equilibrium.

We assume there is a continuum of agents on the unit interval, and in forming expectations they can use one of two models or perceived laws of motion (PLM). Agents in group 1 are "fundamentalists" and use a PLM^1 consistent with the fundamental equilibrium $y = 0$:

$$y_t = y^f \tag{6}$$

where y^f is estimated from data. It follows that

$$E_t^1 y_{t+1} = y^f.$$

Agents in group 2 instead believe to be in a sunspot equilibrium, with PLM^2

$$y_t = \bar{y}_j \;\; \text{if } s_t = s_j,\, j = 1, 2, \tag{7}$$

that is, they condition their forecasts on an observable exogenous variable s_t. Following much of the literature, we assume transition probabilities are known. It follows

$$\begin{aligned} E_t^2 y_{t+1} &= \pi_{11}\bar{y}_1 + (1-\pi_{11})\,\bar{y}_2 \text{ if } s_t = s_1 \\ E_t^2 y_{t+1} &= (1-\pi_{22})\,\bar{y}_1 + \pi_{22}\bar{y}_2 \text{ if } s_t = s_2. \end{aligned}$$

Assuming there is a proportion μ (to be endogenised later) of group 1 agents (and a proportion of $1-\mu$ group 2 agents), aggregate expectations are therefore given by

$$E_t^* y_{t+1} = \mu E_t^1 y_{t+1} + (1-\mu)\, E_t^2 y_{t+1}$$

and the ensuing actual law of motion (ALM) for the economy is

$$\begin{aligned} y_t &= \beta\mu y^f + \beta\,(1-\mu)\,[\pi_{11}\bar{y}_1 + (1-\pi_{11})\,\bar{y}_2] \text{ if } s_t = s_1 \\ y_t &= \beta\mu y^f + \beta\,(1-\mu)\,[(1-\pi_{22})\,\bar{y}_1 + \pi_{22}\bar{y}_2] \text{ if } s_t = s_2. \end{aligned}$$

In order for PLM^i, $i = 1, 2$, to be consistent with the ALM, we need the following sets of restrictions.

Agents using PLM^1 will try to minimize the mean squared error from their predictor, i.e., they estimate their PLM with least-squares techniques. We denote by T_1 and T_2, respectively the time spent in state 1 and state 2 from the stationary distribution associated with Π. The stationary distribution $T = [T_1, T_2]$ is the normalized (i.e., the sum of its entries is equal to 1) left eigenvector of the transition matrix associated with the eigenvalue equal to 1, such that $T = \pi T$, and is given by

$$\begin{aligned} T_1 &= \frac{1-\pi_{22}}{2-\pi_{11}-\pi_{22}} \\ T_2 &= \frac{1-\pi_{11}}{2-\pi_{11}-\pi_{22}}. \end{aligned}$$

Asymptotically, the least-squares estimate for y^f will converge to the weighted average of the two ALMs across states, i.e.,

$$y^f = T_1 \left[\beta\mu y^f + \beta(1-\mu)\left[\pi_{11}\bar{y}_1 + (1-\pi_{11})\bar{y}_2\right]\right] + T_2\left[\beta\mu y^f + \beta(1-\mu)\left[(1-\pi_{22})\bar{y}_1 + \pi_{22}\bar{y}_2\right]\right] \tag{8}$$

or

$$y^f = \beta\mu y^f + \beta(1-\mu)\left[T_1\left(\pi_{11}\bar{y}_1 + (1-\pi_{11})\bar{y}_2\right) + T_2\left((1-\pi_{22})\bar{y}_1 + \pi_{22}\bar{y}_2\right)\right]. \tag{9}$$

Restrictions for PLM^2, such that it also minimizes the mean squared error of predictions, can instead be found by mapping the PLM into the ALM in each state, obtaining

$$\bar{y}_1 = \beta\mu y^f + \beta(1-\mu)\left[\pi_{11}\bar{y}_1 + (1-\pi_{11})\bar{y}_2\right] \text{ if } s_t = s_1, \tag{10}$$

$$\bar{y}_2 = \beta\mu y^f + \beta(1-\mu)\left[(1-\pi_{22})\bar{y}_1 + \pi_{22}\bar{y}_2\right] \text{ if } s_t = s_2. \tag{11}$$

We therefore have three equations, (9), (10) and (11) which jointly determine y^f, $\bar{y}_1$ and $\bar{y}_2$. Clearly the three equations determine a linear homogeneous system in the three unknowns, whose solution is either the zero vector or indeterminate. Only in this last case a sunspot solution exists. Denoting this system

$$Hy = 0 \tag{12}$$

where

$$\begin{aligned} y' &= \left[y^f, \bar{y}_1, \bar{y}_2\right] \\ H &= \begin{bmatrix} \beta\mu - 1 & \beta(1-\mu)\left(T_1\pi_{11} + T_2(1-\pi_{22})\right) & \beta(1-\mu)\left(T_1(1-\pi_{11}) + T_2\pi_{22}\right) \\ \beta\mu & \beta(1-\mu)\pi_{11} - 1 & \beta(1-\mu)(1-\pi_{11}) \\ \beta\mu & \beta(1-\mu)(1-\pi_{22}) & \beta(1-\mu)\pi_{22} - 1 \end{bmatrix}, \end{aligned} \tag{13}$$

we therefore need one eigenvalue of H to be equal to zero.

Noting that $T_1\pi_{11} + T_2(1-\pi_{22}) = T_1$ and $T_1(1-\pi_{11}) + T_2\pi_{22} = T_2$, we can rewrite H as

$$H = \begin{bmatrix} \beta\mu - 1 & \beta(1-\mu)T_1 & \beta(1-\mu)T_2 \\ \beta\mu & \beta(1-\mu)\pi_{11} - 1 & \beta(1-\mu)(1-\pi_{11}) \\ \beta\mu & \beta(1-\mu)(1-\pi_{22}) & \beta(1-\mu)\pi_{22} - 1 \end{bmatrix}.$$

Given that $T_1 + T_2 = 1$, the three eigenvalues of H are $-\frac{1}{2}\left(-2 + \beta \pm \sqrt{\beta^2}\right)$ and $-1 + \beta(1-\mu)(\pi_{11} + \pi_{22} - 1)$. The first two reduce to -1 and $-1+\beta$ and therefore, for generic $\beta \neq 1$, we need the condition

$$\pi_{11} + \pi_{22} = 1 + \frac{1}{\beta(1-\mu)} \tag{14}$$

to be satisfied in order to have a non-zero solution for vector y.

Note that this condition poses a restriction between transition probabilities π_{11} and π_{22}, β and μ. This is the equivalent, in the heterogeneous setting, of the resonant condition found in the literature for the homogeneous case. In fact, for $\mu = 0$, our condition (14) reduces exactly to the one for the homogeneous case: $\pi_{11} + \pi_{22} = 1 + \beta^{-1}$.

Transition probabilities are therefore restricted by β and μ. Note that this imposes a restriction on the admissible values for β and μ, as we must have $0 \leq \pi_{11}, \pi_{22} \leq 1$. This is similar to what happens in the homogeneous case, where bounds on transition probabilities pose a restriction on β. If we impose the restriction $0 \leq \pi_{11} + \pi_{22} \leq 2$ (necessary, but not sufficient, for transition probabilities to be well defined), we have in fact that in the homogeneous case the restriction is $\beta > 1$ or $\beta < -1$, while in our heterogeneous setting the restriction is $\beta > \frac{1}{1-\mu}$ or $\beta < -\frac{1}{1-\mu}$. As $\mu \to 0$, the two sets of conditions coincide.

The vector y is then the eigenvector of the zero eigenvalue associated with matrix H.[1] Note that $y^f = 0$, since $T_1\bar{y}_1 + T_2\bar{y}_2 = 0$: this is consistent with agents in group 1 believing to be in a fundamental equilibrium.

Definition 1 *A 2-states heterogeneous sunspot equilibrium is an endogenous stochastic process y_t, an exogenous 2-state Markov process s_t with a set of transition probabilities $\{\pi_{11}, \pi_{22}\}$ and a set of values $\{y^f, \bar{y}_1, \bar{y}_2\}$, and a population fraction $\mu \in (0,1)$ such that: i) y_t solves (1) for any t; ii) expectations are formed according to (6) and (7); iii) expectations deliver fixed points of the maps (9), (10), (11).*

2.1.1 A numerical example

As an example, we consider an economy represented by the following parameter values: $\beta = 1.5$, $\mu = .1$, $\pi_{11} = .9$, $\pi_{22} = .8407$. Note that this parameterization satisfies the resonant condition outlined above. Such values mean that there is 10% of the population using the "fundamental" PLM and 90% of agents using instead PLM^2 and conditioning their forecast on a sunspot with transition probabilities pinned down by $\pi_{11} = .9$ and $\pi_{22} = .8407$.

From these values, we can compute $T_1 = 0.6143$ and $T_2 = 0.3857$, and $[y^f, \bar{y}_1, \bar{y}_2] = [0, 1, -1.5926]$. In steady state, the economy will be around 61% of the time in state 1 and 39% in state 2, where the value of y will be, respectively 1 and -1.5926: this means that the weighted average of y in steady state is 0, consistent with beliefs of agents in group 1.

3 Equilibria with sunspots in autoregressive form

While early literature considered mainly Markov state sunspot equilibria, in macroeconomics sunspot solutions in ARMA form are more common. It is now known, though, that only when the sunspot has an autoregressive form, such solutions can be learnable (Evans and McGough, 2005a). We will thus focus on such form here.

In the previous section we have followed the literature and carried out the analysis of Markov state sunspot equilibria in a purely forward looking model. While the same model has been used

[1] Since this vector is not uniquely defined, some normalization will be necessary (such as setting $\bar{y}_1 = 1$).

also to analyze equilibria with sunspots in AR(1) form (see, for example, Evans and Honkapohja 2003, Evans and McGough 2011), a more general framework, with stochastic shock and lagged endogenous variable, has also been used (see, for example, Evans and McGough 2005a). We will adopt this more general framework here:

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t, \tag{15}$$

where the exogenous shock v_t is white noise with variance σ_v^2. Of course, equation (15) can be reduced to (1) by setting $\delta = 0$, $v_t \equiv 0$.

Under rational expectations, with $E^* = E$, the expectational operator, equation (15) has one unique non-explosive solution if $0 <\mid \phi_1 \mid < 1 <\mid \phi_2 \mid$, with ϕ_1 and ϕ_2 representing the two roots of the polynomial $\beta\phi^2 - \phi + \delta$:

$$\phi_{1,2} = \frac{1 \pm \sqrt{1 - 4\beta\delta}}{2\beta},$$

with ϕ_1 being the root obtained with the minus sign. In this case the unique solution takes the form[2]

$$y_t = \phi_1 y_{t-1} + (\beta\phi_2)^{-1} v_t. \tag{16}$$

We will be interested here instead in the case with $0 <\mid \phi_1 \mid<\mid \phi_2 \mid< 1$, as in this case solutions other than the minimum state variable (MSV) one exist,[3] and sunspots can play a role in the model.

To fix ideas, consider the general solution to model (15), that can be written as

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1}\delta y_{t-2} - \beta^{-1} v_{t-1} + \varepsilon_t \tag{17}$$

where

$$\varepsilon_{t+1} = y_{t+1} - E_t y_{t+1}$$

is a martingale difference sequence (mds). By defining

$$\varepsilon_t = (\beta\phi_i)^{-1} v_t,$$

$i \in \{1, 2\}$, we get the two MSV solutions: depending on the case, neither, one or both solutions can be stable.

If instead

$$\varepsilon_t = (\beta\phi_i)^{-1} v_t + (1 - \phi_i L)\, \xi_t,$$

we have a sunspot solution, where ξ_t is the sunspot component, independent of v_t.[4] This last equation imposes a restriction on the sunspot, which must "resonate" with the structural parameters for the economy, i.e., it must have an AR(1) form with coefficient equal to ϕ_i (see Evans and McGough 2005a).

[2] We will restrict our analysis to the case where these roots are real, which imposes the restriction $\beta\delta < 1/4$.

[3] The term "minimum state variables" was introduced by McCallum (1983) and refers to the solution with the minimum possible number of state variables.

[4] For a derivation of the MSV and sunspot solutions from (17), see the Appendix 7.1.

Formally, for a given sunspot variable

$$\xi_t = \lambda \xi_{t-1} + \varepsilon_t$$

we must have that

$$\lambda = \phi_i$$

in order for the sunspot to enter into the solution of (15).

Note that the sunspot solution imposes a restriction on the roots ϕ_i: they both need to be smaller than one in absolute value in order to have a stable process for y_t.

It is well known that, for a given solution, we can have alternative representations, as it has been shown by Evans and McGough (2005a, 2005b). Only sunspot solutions with the so called common factor (CF) representation, though, turn out to be learnable. We will thus focus on these solutions in our analysis, and in particular on the only one being learnable:

$$y_t = \phi_1 y_{t-1} + (\beta\phi_2)^{-1} v_t + \kappa \xi_t, \tag{18}$$

with

$$\xi_t = \phi_2 \xi_{t-1} + \varepsilon_t.$$

Existence and E-stability of this solution requires $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.[5] In this case, moreover, the parameter κ attached to the sunspot is free. Note that these conditions imply $\phi_{1,2} < 0$.

3.1 Heterogeneous solutions

Starting from the results of Evans and McGough (2005a), outlined in the previous section, we want now to introduce heterogeneity of beliefs in this framework and investigate whether solutions exist where only a fraction of agents conditions their forecasts on the sunspot. We therefore allow agents to use one of two different models, one that includes and one that does not include a sunspot component: while the fraction of agents in each group is fixed in this section, it will be endogenized later using evolutionary dynamics, where we will allow agents to choose between different models on the basis of the relative performance in forecasting, as expressed by the expected mean square errors (MSE). In specific, we will use replicator dynamics to model the evolution of the fraction of agents using each model.

The economy is still represented by (15). We assume there is a continuum of agents on the unit interval, and in forming their expectations they can use one of two models or $PLMs$, one sunspot-free (PLM^1)

$$y_t = a_1 + b_1 y_{t-1} + c_1 v_t \tag{19}$$

and one that includes a sunspot (PLM^2)

$$y_t = a_2 + b_2 y_{t-1} + c_2 v_t + d_2 \xi_t. \tag{20}$$

[5] These restrictions are obtained by Evans and McGough (2005a) by simultaneously imposing conditions for indeterminacy, E-stability and real solution. Conditions for E-stability alone would be $\beta < 1/2$ or $\beta + \delta < 1$.

Parameters in each model will be derived by minimizing the expected mean squared errors of forecasts, which is equivalent to saying that they are estimated using least-squares techniques.

We denote by μ the fraction of agents using the sunspot-free model, and $(1-\mu)$ the remaining agents using the model with sunspot. Aggregate expectations are therefore given by

$$E_t^* y_{t+1} = \mu E_t^1 y_{t+1} + (1-\mu) E_t^2 y_{t+1}$$

where $E_t^i y_{t+1}$ are expectations formed using PLM^i, $i \in \{1, 2\}$.

The parameter μ, for the moment taken as given, will be regarded later on as endogenous, and it will be determined by replicator dynamics based on the relative performance of the two models, measured by the unconditional mean squared error.[6]

One important thing that must be noted is that the presence of heterogeneity changes the resonant frequency condition for the sunspot, as already pointed out by Berardi (2009) in a purely forward looking model. To see how this happens in the present context, and to fix ideas about learning, consider agents using (19) and (20) to form expectations. We then have

$$\begin{aligned} E_t^1 y_{t+1} &= a_1(1+b_1) + b_1^2 y_{t-1} + c_1 b_1 v_t && (21) \\ E_t^2 y_{t+1} &= a_2(1+b_2) + b_2^2 y_{t-1} + c_2 b_2 v_t + d_2(b_2+\lambda)\xi_t, && (22) \end{aligned}$$

where we have assumed that λ is known to agents: otherwise, since the sunspot component is exogenous and observable, this parameter could be consistently estimated using least-squares techniques. Aggregating expectations and substituting into (15), we get the temporary equilibrium, or ALM for the economy:

$$\begin{aligned} y_t &= \beta\left[\mu a_1(1+b_1) + (1-\mu)a_2(1+b_2)\right] + \left[\beta\left(\mu b_1^2 + (1-\mu)b_2^2\right) + \delta\right] y_{t-1} + \\ &\quad + \left[\beta\left(\mu c_1 b_1 + (1-\mu)c_2 b_2\right) + 1\right] v_t + \beta\left[(1-\mu)d_2(b_2+\lambda)\right]\xi_t. && (23) \end{aligned}$$

Equilibrium value for parameters under learning are those that minimize the mean squared errors of forecasts. While for PLM^2 we can obtain such values by simply mapping one by one parameters from PLM^2 to ALM, this is not possible for PLM^1. This last one is in fact under-specified with respect to the ALM, since it ignores the sunspot component that enters in the ALM through the expectations of agents in group 2. Parameters from PLM^1 can therefore not be mapped directly into those of the ALM: the best thing agents can do is to optimize parameters in their PLM^1 in a statistical sense, by minimizing the expected mean squared errors, and to find such values we must project the parameters of the ALM into the space of those in the PLM. We thus obtain that parameters are defined by the maps (see Appendix 7.2):

[6] As Branch and Evans (2006) point out, the unconditional mean square error is more appropriate than a measure of real time performance in a stochastic framework such as ours.

$$
\begin{aligned}
a_1 &= A && (24)\\
a_2 &= A && (25)\\
b_1 &= B+F && (26)\\
b_2 &= B && (27)\\
c_1 &= C && (28)\\
c_2 &= C && (29)\\
d_2 &= D && (30)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \beta\left[\mu a_1\left(1+b_1\right)+\left(1-\mu\right)a_2\left(1+b_2\right)\right]\\
B &= \beta\left(\mu b_1^2+\left(1-\mu\right)b_2^2\right)+\delta\\
C &= \beta\left[\mu c_1 b_1+\left(1-\mu\right)c_2 b_2\right]+1\\
D &= \beta\left(1-\mu\right)\left(b_2+\lambda\right)d_2\\
F &= \frac{\lambda D^2\left(1-B^2\right)\sigma_\xi^2}{\left(1-\lambda B\right)\left(A^2+C^2\sigma_v^2\right)+\left(1+\lambda B\right)D^2\sigma_\xi^2}.
\end{aligned}
$$

The key difference between the two sets of belief parameters for the two groups comes from F: for agents not using the sunspot, the coefficient b_1 on y_{t-1} must pick up the serial correlation that comes from the sunspot. Note that if $d_2 = 0$, $\sigma_\xi^2 = 0$ or $\lambda = 0$, then $F = 0$ and $b_1 = b_2$. In the first case, the sunspot effectively does not enter into PLM^2, and thus it does not affect the ALM and b_1 does not need to adjust for it; in the second case, the sunspot is equal to zero at all times, and therefore, again, it does not affect the dynamics of y_t; finally, in the third case, the sunspot is an i.i.d. process and therefore it does not induce any additional serial correlation in y_t, so b_1 does not need to adjust to compensate for it.

Fixed points of equations (24)-(30) are possible equilibria for the model: agents are doing the best they can, given the model they are endowed with. Note that agent not conditioning their forecasts on the sunspot have in fact a misspecified model, and their parameter b_1 needs to pick up the correlation induced by the sunspot in order to minimize the mean squared errors. Beliefs derived from PLM^1 imply a restricted perceptions equilibrium (RPE) for those agents (see Evans and Honkapohja, 2001, chap. 13): they represent a weaker requirement compared to what is asked in a REE, since forecasts are assumed to be optimal only within a particular class of (underparameterized) linear models.

We turn now to the map for d_2. In order to have it satisfied, and thus obtain an equilibrium with sunspot, we must have

$$\beta\left(1-\mu\right)\left(b_2+\lambda\right)=1. \tag{31}$$

This is the resonant condition in the heterogeneous setting, and it poses a restriction between β, μ and λ, much in the same way as we found for the 2-SSE. Instead of the transition probabilities, the restriction now affects the AR(1) coefficient on the sunspot process.

Definition 2 *A heterogeneous sunspot equilibrium with AR(1) sunspot is an endogenous stochastic process y_t, a fundamental shock v_t and an exogenous stochastic process ξ_t, with population fraction $\mu \in (0,1)$ and a set of expectational parameters $\{a_1, b_1, c_1, a_2, b_2, c_2, d_2\}$ such that: i) y_t solves (15) for any t; ii) expectations are given by (**??**),(21) and (22); iii) expectational parameters are fixed points of the maps (24)-(30); iv) condition (31) is satisfied and $d_2 \neq 0$.*

Note that we are requiring $\mu \in (0,1)$, i.e., both $PLMs$ must be used in a heterogeneous equilibrium. If instead $\mu \in \{0,1\}$, we then have an homogeneous expectations equilibrium (with or without sunspot, respectively).

The set of maps (24)-(29) plus the resonant fraction condition $\beta[(1-\mu)(b_2+\lambda)] = 1$ define parameters $\{a_1, b_1, c_1, a_2, b_2, c_2\} \cup \{\mu\}$, for given β, δ and λ, while d_2 is free. Note that, for given β, δ and λ, the resonant condition pins down μ. Alternatively, for given β, δ and μ, it pins down λ.

The set of equations (24)-(29) can only be solved numerically. In particular, we can use a multivariate Newton procedure starting from the known, closed form equilibrium of a system with $d_2 = 0$ (i.e., without sunspot): in this case in fact, the solution is given by (16), i.e.,

$$\begin{aligned}
\bar{a}_1 &= \bar{a}_2 = 0 \\
\bar{b}_1 &= \bar{b}_2 = \phi_1 \\
\bar{c}_1 &= \bar{c}_2 = \frac{1}{1-\beta\phi_1}.
\end{aligned}$$

Starting from such values, and using equations (24)-(29) plus (31) and their derivatives, we can compute the values of coefficients in the two $PLMs$. An example is given in the next Section.

3.1.1 A numerical example

In order to implement our numerical procedure to find an heterogeneous equilibrium, we need to chose a set of parameters values. We will use in this example the following set of values: $\beta = -0.8$; $\delta = -0.3$; $\lambda = -0.9$; $\sigma_v^2 = 1$; $\sigma_\xi^2 = .1$; $d_2 = 1$. Note that this parameterization satisfies all the restrictions derived by Evans and McGough (2005a) to ensure E-stability.[7] In fact, such restrictions are quite demanding and only admit a small set of combinations of parameters.

Let's define $\{a_1^*, b_1^*, c_1^*, a_2^*, b_2^*, c_2^*\} \cup \{\mu^*\}$ as the solution vector, such that equations (24)-(29) plus (31) are satisfied. We then find

$$\begin{aligned}
a_1^* &= 0.000000, \quad a_2^* = 0.000000 \\
b_1^* &= -0.559538, \quad b_2^* = -0.521657 \\
c_1^* &= 1.727071, \quad c_2^* = 1.727071 \\
\mu^* &= 0.120744.
\end{aligned}$$

[7] Moreover, with this parameterization, the resonant condition implies an initial value for μ equal to 0.107, which satisfies the restriction $0 < \mu < 1$.

Note that the only difference for the belief coefficients in the two groups of agents (besides d_2, which is free) is in parameters b_1 and b_2, as expected from looking at equations (24)-(29).

4 Evolutionary dynamics

We consider now how a sunspot equilibrium could emerge from a process of evolution and adaptation on agents' beliefs. We therefore look at the possibility of a transition from a fundamental equilibrium to a sunspot one. This is equivalent, in our heterogeneous setting, to going from $\mu = 1$ to $\mu = 0$. Unless the whole population switches simultaneously and instantly agrees to coordinate on a sunspot, this transition will need to happen gradually, through some process of progressive adoption of the sunspot in the population. We will consider how this can be achieved through evolutionary dynamics.

A note here must be made about the nature of time in our model. Though time in equations (1) and (15) is discrete, adaptive learning dynamics can be studied (asymptotically) by analyzing the behavior of ordinary differential equations that represent approximations to the original stochastic recursive algorithm (see Evans and Honkapohja, 2001), and we have exploited this fact in order to derive the equilibrium values of parameters in the $PLMs$. In order to gain analytical tractability, we will also model replicator dynamics in continuous time. The choice of continuous time replicator dynamics in a discrete time model can be justified by noting that such continuous time dynamics are the approximation of the discrete time replicator dynamics as the interval of time is decreased towards zero (see Vega-Redondo 2008, chapter 3). In particular, the continuous time replicator equation can be interpreted as an approximation to a discrete time system where the magnitude of differential fitness per period between types is very small: having the time interval become infinitesimal in the replicator dynamics equation is one way of achieving such approximation. In a later Section, we will consider predictor choice dynamics in discrete time, and show that results do not change.

Under replicator dynamics, the fraction of agents using each model (μ) evolves according to their relative performance. Using the (unconditional) mean squared error (MSE) as a measure of performance, we thus have

$$\begin{aligned} \dot{\mu} &= \mu \left\{ -MSE^1 - \left[\mu(-MSE^1) + (1-\mu)\left(-MSE^2\right) \right] \right\} \\ &= \mu\,(1-\mu)\,\Delta \end{aligned} \tag{32}$$

where $\Delta = MSE^2 - MSE^1$. $MSEs$ will be defined below for each model and setting specification.

Clearly, $\Delta = 0$ implies $\dot{\mu} = 0$, as the two models deliver the same performance and there is no incentive for agents to switch from one to the other. Moreover, also $\mu = 0$ and $\mu = 1$ are resting points for the dynamics of μ: once homogeneity is reached, the excluded model is no longer used. We will check below whether these equilibrium points are stable under replicator dynamics. Results will depend on the sign of the derivative $\frac{\partial \dot{\mu}}{\partial \mu}$ at the two resting points: for $\mu = 0$, $\frac{\partial \dot{\mu}}{\partial \mu} = \Delta$, while for $\mu = 1$, $\frac{\partial \dot{\mu}}{\partial \mu} = -\Delta$.

We will also be interested, in particular, to see whether an equilibrium for the dynamics of the two groups of agents exists other than the two homogeneous ones, i.e., if there exists a situation where $\dot{\mu} = 0$ but $\mu \notin \{0, 1\}$: this will require $\Delta = 0$.

Definition 3 *An endogenous heterogeneous expectations equilibrium with sunspot, either in 2-state Markov or AR(1) from, is a heterogeneous expectations equilibrium as defined in Definition (1) or in Definition (2), but where the population fraction $\mu \in (0, 1)$ is endogenously determined by (32).*

4.1 Replicator dynamics in 2-SSE

In order to have the relative fraction of agents using each model (μ) determined endogenously through replicator dynamics, as specified by equation (32), we need to derive the unconditional expected MSE for the two models:

$$\begin{aligned} MSE^1 &= E\left(y_t - y^f\right)^2 \\ MSE^2 &= E\left(y_t - \bar{y}_1\right)^2 \text{ if } s_t = s_1 \\ MSE^2 &= E\left(y_t - \bar{y}_2\right)^2 \text{ if } s_t = s_2. \end{aligned}$$

Substituting the ALM into these expressions, we get:

$$MSE^1 = E\left[\beta\mu y^f + \beta\left(1-\mu\right)\left[\pi_{11}\bar{y}_1 + \pi_{22}\bar{y}_2\right] - y^f\right]^2 \text{ if } s_t = s_1, s_2$$

and

$$\begin{aligned} MSE^2 &= E\left[\beta\mu y^f + \beta\left(1-\mu\right)\left[\pi_{11}\bar{y}_1 + \pi_{22}\bar{y}_2\right] - \bar{y}_1\right]^2 \text{ if } s_t = s_1 \\ MSE^2 &= E\left[\beta\mu y^f + \beta\left(1-\mu\right)\left[\pi_{11}\bar{y}_1 + \pi_{22}\bar{y}_2\right] - \bar{y}_2\right]^2 \text{ if } s_t = s_2. \end{aligned}$$

Looking at the maps from $PLMs$ to ALM, it is clear that $MSE^2 = 0$ at all times, while $MSE^1 > 0$.

Specifically, we have that if $s_t = s_1$

$$\Delta = MSE^2 - MSE^1 = -MSE^1 = -E\left[\left(\beta\mu - 1\right)y^f + \left[\beta\left(1-\mu\right)\pi_{11} - 1\right]\bar{y}_1 + \beta\left(1-\mu\right)\pi_{22}\bar{y}_2\right]^2$$

and if $s_t = s_2$

$$\Delta = MSE^2 - MSE^1 = -MSE^1 = -E\left[\left(\beta\mu - 1\right)y^f + \beta\left(1-\mu\right)\pi_{11}\bar{y}_1 + \left(\beta\left(1-\mu\right)\pi_{22} - 1\right)\bar{y}_2\right]^2.$$

Since in both cases $MSE^1 > 0$, we will have $\Delta = MSE^2 - MSE^1 < 0$ and $\mu \to 0$ over time. It follows that under replicator dynamics there can be no equilibrium where $\mu \in (0, 1)$.

Proposition 4 *An endogenous heterogeneous expectations equilibrium with 2-state Markov sunspot does not exist.*

Consider the problem of stability under evolutionary dynamics now: of the two homogeneous equilibria represented by $\mu = \{0, 1\}$, only the first one is stable, since at $\mu = 0$ we have $\Delta < 0$ and thus $\frac{\partial \dot{\mu}}{\partial \mu} = \Delta < 0$. The second one is instead unstable, since at $\mu = 1$, $\Delta < 0$, and thus

$\frac{\partial \dot{\mu}}{\partial \mu} = -\Delta > 0$: a small deviation from the homogeneous fundamental equilibrium is enough to drive the economy away from that equilibrium.

Proposition 5 *The homogeneous equilibrium with $\mu = 0$ (i.e., the sunspot equilibrium) is always stable under replicator dynamics. The homogeneous equilibrium with $\mu = 1$ (i.e., the fundamental equilibrium) is never stable under replicator dynamics.*

Our results above seem to suggest that, starting from a homogeneous fundamental equilibrium, the economy would move towards a homogeneous sunspot equilibrium as soon as a fraction of agents start using the sunspot for their forecasts. But for this to be true, the resonant condition should hold at all times over the path of the evolutionary dynamics if the two $PLMs$ are to represent an equilibrium for agents' beliefs. Since the replicator dynamics equation $\dot{\mu} = \mu(1-\mu)\Delta$ implies that μ changes for any $\Delta \neq 0$ (in particular, $\Delta < 0$), condition (14) should then adjust accordingly.

Consider the transition probabilities defining the sunspot, π_{11} and π_{22}. We have seen that they are pinned down by β and μ through the resonant condition. But under evolutionary dynamics, μ changes according to (32). Since β is a deep parameter unlikely to adjust as μ moves under evolutionary dynamics, the (exogenous) transition probabilities should adjust to ensure the resonant condition is satisfied and the sunspot equilibrium exists all along the evolutionary trajectories. This is clearly not possible, since the sunspot is an exogenous process and there is no reason it should evolve in line with μ.

There is of course another possibility, that is, as μ evolves under replicator dynamics, agents could continuously change the sunspot variable they are conditioning on, in order to ensure that the resonant condition remains satisfied at all times. This would require the existence of a continuum of sunspots with the right statistical properties, and agents knowingly moving from one to another as population dynamics evolve: something that seems quite farfetched to assume.

Proposition 6 *No exogenous two state Markov process can represent a sunspot variable along evolutionary trajectories, as the resonant condition (14) will be continually violated due to the dynamics in μ.*

What would happen when, along evolutionary trajectories, the resonant condition is violated? Matrix H from (13) would not be singular, and the only solution for the system (12) would be $\left[y^f, \bar{y}_1, \bar{y}_2\right] = [0,0,0]$: the fundamental equilibrium. It follows that we can not obtain a Markov sunspot equilibrium through a process of learning and evolution on agents' beliefs.

4.2 Replicator dynamics in AR(1) sunspot equilibrium

We turn now to the case with AR(1) sunspot and allow the relative fraction of agents using each model (μ) to be determined endogenously through replicator dynamics, as specified by equation (32). We start by defining the unconditional expected MSE for the two PLMs in this setting as:

$$\begin{aligned} MSE^1 &= E\left(y_t - \theta_1 z_t\right)^2 \\ MSE^2 &= E\left(y_t - \theta_2 z_t\right)^2 \end{aligned}$$

where

$$\begin{aligned}\theta_1 &= [a_1^* \; b_1^* \; c_1^* \; 0] \\ \theta_2 &= [a_2^* \; b_2^* \; c_2^* \; d_2^*]\end{aligned}$$

and

$$z_t = [1 \; y_{t-1} \; v_t \; \xi_t]'.$$

Remember that in order to have $\dot{\mu} = 0$, we need either $\mu \in \{0, 1\}$, or $\Delta = 0$.

Using the maps from PLMs to ALM, we obtain

$$\begin{aligned}MSE^1 &= E\left(A - a_1 + (B - b_1)\, y_{t-1} + (C - c_1)\, v_t + D\xi_t\right)^2 = \left(-Hy_{t-1} + D\xi_t\right)^2 > 0 \\ MSE^2 &= E\left(A - a_2 + (B - b_2)\, y_{t-1} + (C - c_2)\, v_t + (D - d_2)\, \xi_t\right)^2 = 0\end{aligned}$$

and therefore

$$\Delta = MSE^2 - MSE^1 = -\left(-Hy_{t-1} + D\xi_t\right)^2 < 0$$

This result shows that the requirement for a heterogeneous equilibrium where $\mu \notin \{0, 1\}$ can not be met, as agents using the sunspot will always outperform the others in terms of $MSEs$. Under replicator dynamics, there does not exist an equilibrium where both types of agents coexist.

Proposition 7 *An endogenous heterogeneous expectations equilibrium with AR(1) sunspot does not exist.*

We want now to check for stability under replicator dynamics of the two homogeneous solutions, $\mu = 0$ and $\mu = 1$. Note that both imply $\dot{\mu} = 0$, so they are resting points of the population dynamics, but are they locally stable? To answer this question we can simply observe that $\Delta < 0$, for any $\mu \in (0, 1)$, which implies that $\dot{\mu} < 0$: so while $\mu = 1$ is unstable under replicator dynamics, $\mu = 0$ is stable. Even if a small fraction of agents starts using the sunspot, evolutionary dynamics ensure that all the population gradually switches to the sunspot. Another way of looking at it is to check for the sign of the derivative $\frac{\partial \dot{\mu}}{\partial \mu}$ at the two resting points: we have that for $\mu = 0$, $\frac{\partial \dot{\mu}}{\partial \mu} = \Delta < 0$, while for $\mu = 1$, $\frac{\partial \dot{\mu}}{\partial \mu} = -\Delta > 0$: the first point is therefore stable, while the second is not.

Proposition 8 *The homogeneous equilibrium with $\mu = 0$ (i.e., the sunspot equilibrium) is always stable under replicator dynamics. The homogeneous equilibrium with $\mu = 1$ (i.e., the fundamental equilibrium) is never stable under replicator dynamics.*

This result would seem to support the view that, starting from a homogeneous fundamental equilibrium, we could move to a homogeneous sunspot equilibrium: it only takes a small fractions of agents to use the sunspot for starting the evolutionary dynamics that ultimately lead all agents adopting the sunspot in their forecasts. But the above propositions only considers replicator dynamics, and we have seen that in order to have a sunspot equilibrium with (boundedly[8]) rational

[8] We qualify the word rational with boundedly here becasue agents that do not condition their forecast are using a mispecified model. But given that model, they are doing the best they can, i.e., their model is optimal in terms of delivering the minimal mean squared error possible.

agents, a resonant condition must be satisfied.

This condition, for given β, δ and λ, pins down a specific value for μ. For any other value of μ, the only equilibrium value for d_2 from (30) would be zero, thus effectively eliminating the impact of the sunspot in the economy. This means that as μ changes under evolutionary dynamics, the resonant condition will be violated and the sunspot equilibrium disappear.

Alternatively, for given β, δ and μ, the resonant condition imposes a restriction on λ. For example, for $\mu = 0$, the resonant condition requires $\beta\left[(b_2 + \lambda)\right] = 1$, i.e.,

$$\lambda = \beta^{-1} - b_2.$$

If an exogenous stochastic process with such AR(1) coefficient exists, it could be the basis for a sunspot equilibrium where all agents condition they forecasts on it. In a heterogeneous setting with only a fraction $\mu \in (0,1)$ of agents using the sunspot, the resonant condition is instead

$$\beta\left(1-\mu\right)\left(b_2+\lambda\right) = 1$$

or

$$\lambda = \left[\beta\left(1-\mu\right)\right]^{-1} - b_2$$

where b_2 is pinned down by the optimality condition on belief parameters, and it depends on β, δ and μ. Therefore, since β and δ are taken to be structural and fixed parameters, as μ changes under evolutionary dynamics, λ would need to change too in order for the restriction to remain true and the sunspot equilibrium to exist. This means that along evolutionary trajectories, in order for the resonant condition to remain satisfied and the sunspot equilibrium to exist, either the sunspot endogenously modifies its (exogenous) statistical properties, or agents continuously change the sunspot variable on which they condition their forecasts. Either case implies that a given exogenous stochastic process can not represent a sunspot variable when evolutionary dynamics on predictor selection are at play.

Proposition 9 *No exogenous AR(1) stochastic process can represent a sunspot variable along evolutionary trajectories, as the resonant condition (31) will be continually violated due to the dynamics in μ.*

5 Alternative predictor choice dynamics

We have used replicator dynamics to model the evolution of predictor choices over time in the population. Our main argument against the likelihood of sunspot equilibria to emerge from evolutionary dynamics, though, can be made with any evolutionary model of belief dynamics. As an example, we consider here the discrete-time predictor choice dynamics proposed by Brock and Hommes (1997). Brock and Hommes (1997) propose to model the evolution of predictor choices in the population according to a discrete logit model based on relative performance. Using again the unconditional mean squared errors, we have

$$\mu_{t+1} = \frac{\exp\{-\beta MSE_t^2\}}{\exp\{-\beta MSE_t^1\} + \exp\{-\beta MSE_t^1\}} \tag{33}$$

or equivalently

$$\mu_{t+1} = \frac{1}{2}\left(\tanh\left[\frac{\beta}{2}(MSE_t^2 - MSE_t^1)\right] + 1\right), \tag{34}$$

where β is the "intensity of choice" parameter, a measure of agents' "rationality": with $\beta \to \infty$, all agents switch immediately to the predictor that delivers the lowest MSE, while for finite values of β, some agents adopt the sub-optimal predictor. We have added the time t subscript to $MSEs$ to make it explicit that they depend on time t parameters (and in particular on μ_t). Both replicator dynamics and the Brock and Hommes dynamics with $\beta \to \infty$ ensure that no agent will use strictly dominated predictors in equilibrium.

We can see that, since we have $\Delta = MSE^2 - MSE^1 < 0$, in the limiting case $\beta \to \infty$ all agents will ultimately adopt the predictor that includes the sunspot and $\mu \to 0$, both for the sunspots in finite state Markov form and in AR(1) form. Moreover, and crucially for our argument, also in this case the changing fraction of agents using the sunspot requires a time-varying resonant condition (in discrete steps now instead of continuously), thus again calling into question the possibility of a sunspot equilibrium to emerge through evolutionary dynamics in the population.

6 Conclusions

There are two main sources of belief dynamics in our setting: optimization of parameters within each model through learning, and evolutionary dynamics on the selection of the forecasting model.

It has been shown in the literature (in homogeneous settings) that, depending on the parameterization of the model, sunspot equilibria may or may not be learnable: that is, if all agents use the same sunspot in their forecasting model, they may converge to a sunspot equilibrium, or learn to discard the sunspot variable and converge to the fundamental equilibrium.

We have shown here instead that evolutionary dynamics always favour the model with the sunspot, as it always delivers a better performance in the mean squared error sense compared to a model without the sunspot.

While learning dynamics are driven by the difference (in terms of forecasting error) between the ALM and each PLM, evolutionary dynamics are driven by the difference between the (two) competing $PLMs$. In the first case, if the impact of the sunspot variable on the endogenous variable in the ALM is less than what predicted from the PLM (because it enters into the law of motion only indirectly, though expectations, and not directly through the structural model as the fundamental variables), agents can learn to discard the sunspot. In the second case, instead, since the PLM that disregards the sunspot is, effectively, mispecified as long as there is a fraction of agents conditioning on the sunspot, this "fundamental" PLM is bound to deliver higher MSEs, and it will vanish from population over time.

As we have seen, though, evolutionary dynamics clash with the resonant condition necessary for the existence of a sunspot equilibrium: as agents progressively switch from one model to the other, the statistical properties of the sunspot (it's transition probabilities, in case of a 2-state Markov process, or it's serial correlation in case of an AR(1) process) need to adapt endogenously for the sunspot equilibrium to survive. As this can not happen with exogenous stochastic processes, the sunspot will not resonate with the structure of the economy, and agents will stop using it in their forecasts.

The key point in this result is that, in a heterogeneous setting, the ALM for the economy depends on the fraction of agents using the sunspot, μ. The resonant condition that makes the $PLMs$ consistent with the ALM and allows for a sunspot equilibrium depends therefore, among other things, on μ and it imposes a restriction on the relation between such parameter and the statistical properties of the sunspot (π or λ). Under evolutionary dynamics on predictor selection, μ changes endogenously over time: if the equilibrium is to exist at all times over the evolutionary dynamics path, the properties of the sunspot (π or λ) must evolve together with population dynamics. This effectively means that either the changing fraction of population must continuously coordinate on a different sunspot, or the stochastic properties of the (exogenous) sunspot must evolve together with population dynamics. Either case implies that a (given and exogenous) stochastic process cannot represent a sunspot variable over paths defined by evolutionary dynamics and we therefore conclude that sunspot equilibria can not emerge endogenously through a process of learning and adaptation on beliefs. For a sunspot equilibrium to emerge, all agents, simultaneously, must start using the right (i.e., one that has the required statistical properties) sunspot variable.

7 Appendix

7.1 Rational expectations solutions for (15)

We briefly show here how to derive the MSV and sunspot solutions for model (15). Starting from the general form (17), reproduced here for simplicity

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1} \delta y_{t-2} - \beta^{-1} v_{t-1} + \varepsilon_t \tag{35}$$

different solutions can be obtained by appropriately redefining the error term

$$\varepsilon_{t+1} = y_{t+1} - E_t y_{t+1}.$$

1. MSV solutions. By defining

   $$\varepsilon_t = (\beta \phi_i)^{-1} v_t,$$

   $i \in \{1, 2\}$, and deleting the common factor $(1 - \phi_i L)$, we get the two MSV solutions

   $$y_t = \phi_j y_{t-1} + (\beta \phi_i)^{-1} v_t \tag{36}$$

   where $i, j \in \{1, 2\}$ and $j \neq i$. Depending on the case, these two solutions can be both stable, one stable and one not, or both unstable.

2. Sunspot solutions. By defining

   $$\varepsilon_t = (\beta \phi_i)^{-1} v_t + (1 - \phi_i L)\, \xi_t,$$

and again deleting the common factor $(1-\phi_i L)$, we have the sunspot solutions

$$y_t = \phi_j y_{t-1} + (\beta\phi_i)^{-1} v_t + \xi_t, \tag{37}$$

where ξ_t is the sunspot component and again $i, j \in \{1,2\}$, $j \neq i$. Note that in order to have a stable solution for y_t, it must be that $0 <\mid \phi_1 \mid<\mid \phi_2 \mid< 1$, as this requires that both ϕ_i and ϕ_j are less than one in absolute value: the first is needed to ensure that the sunspot is a stable process (if not, it would make unstable also y_t), and the second is required to satisfy the usual stability restriction for the AR(1) coefficient on y_t in (37).

Solution representations (36) and (37) are called "common factor representation", as they are obtained from the general solution (35) by deleting a common factor component.

7.2 Derivation of optimal values for belief parameters

We derive here the value of belief parameters in $PLMs$ that minimize the mean squared errors. We will follow the learning literature and use stochastic approximation to project the ALM onto the space of the PLM for each agent. For agents with a correctly specified model, this is equivalent to map, one by one, parameters from the PLM into the corresponding ones in the ALM. For agents having a misspecified (underparameterized) model, the derivation of the map is more involved and is shown here.

For agents in group 2 (those using the sunspot), the one-to-one map from PLM to ALM leads to equations (25),(27),(29),(30).

For agents in group 1, instead, we must project the ALM onto the space of parameters in the PLM, in order to find their optimal values. We follow the adaptive learning literature here and assume agents recurrently estimate their PLM over time using a recursive least-squares algorithm of the form

$$\begin{aligned}
\phi_t &= \phi_{t-1} + t^{-1} R_t^{-1} x_{t-1}\left(y_t - \phi_{t-1}' x_{t-1}\right) \\
R_t &= R_{t-1} + t^{-1}\left(x_{t-1}x_{t-1}' - R_{t-1}\right)
\end{aligned}$$

where $\phi_t' = [a_1, b_1, c_1]$ and $x_{t-1}' = [1, y_{t-1}, v_t]$.

The E-stability principle allows us to study the asymptotic behavior of this algorithm by looking at the associated ODE arising from stochastic approximation of the discrete time algorithm

$$\frac{d\phi}{d\tau} = \lim_{t\to\infty} E R_t^{-1} x_{t-1}\left(y_t - \phi_{t-1}' x_{t-1}\right)$$

where the expectation is taken over the invariant distribution of x for fixed ϕ.

Since $lim_{t\to\infty} R_t = Exx := R$, where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix},$$

with $\sigma_y^2 \equiv E y_t^2$ and $\sigma_v^2 \equiv E v_t^2$, we then have

$$\frac{d\phi}{d\tau} = RR^{-1}\begin{bmatrix} A-a_1 \\ B-b_1 \\ C-c_1 \end{bmatrix} + R^{-1}DE\xi_t\begin{pmatrix} 1 \\ y_{t-1} \\ v_t \end{pmatrix}$$

$$\frac{d\phi}{d\tau} = \begin{bmatrix} A-a_1 \\ B-b_1 \\ C-c_1 \end{bmatrix} + R^{-1}D\begin{bmatrix} 0 \\ E\xi_t y_{t-1} \\ 0 \end{bmatrix}.$$

and since $E\xi_t = 0$ and $E(\xi_t v_t) = 0$ we get

$$\frac{d\phi}{d\tau} = \begin{bmatrix} A-a_1 \\ B-b_1+D\frac{E\xi_t y_{t-1}}{\sigma_y^2} \\ C-c_1 \end{bmatrix}$$

with

$$\sigma_y^2 = \left(1-B^2\right)^{-1}\left(A^2+2BD\right)\sigma_{\xi y}^2 + C^2\sigma_v^2 + D^2\sigma_\xi^2$$

$$\sigma_{\xi y}^2 \equiv E\xi_t y_{t-1} = \frac{D\lambda}{1-B\lambda}\sigma_\xi^2,$$

which lead to equations (24),(26),(28).

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