Discussion Paper Series

Heterogeneous sunspots solutions under learning and replicator dynamics

By

Michele Berardi

Centre for Growth and Business Cycle Research, Economic Studies,
University of Manchester, Manchester, M13 9PL, UK

September 2011
Number 160

Heterogeneous sunspots solutions under learning and replicator dynamics

Michele Berardi
The University of Manchester

September 7, 2011

Abstract

In a linear stochastic forward-looking univariate model with predetermined variables, we consider the possibility of heterogeneous equilibria with sunspots emerging endogenously through adaptive learning and replicator dynamics. In particular, we investigate equilibria where only a fraction of agents in the economy condition their forecasts on a sunspot, and equilibria where different groups of agents use different sunspots. We find that, although such heterogeneous equilibria exist and can be stable under adaptive learning, they do no survive under endogenous replicator dynamics. Moreover, we show that even homogeneous sunspot equilibria require some degree of coordinations among agents for them to emerge in an economy. We conclude that heterogeneous equilibria with sunspots are fragile under endogenous selection of predictors by agents, and that even the relevance of homogeneous sunspot equilibria is questioned once agents are allowed to doubt about the importance of sunspots in their forecasts.

Key words: Sunspots; heterogeneity; expectations; learning.

JEL classification: C62, D83, D84, E32.

Heterogeneous sunspots solutions under learning and replicator dynamics

1 Introduction

Sunspot equilibria are an intriguing possibility, as they open the door to fluctuations in economic activity driven purely by agents' expectations and disconnected from economic fundamentals. From the seminal works of Azariadis (1981) and Cass and Shell (1983), the possibility of self-fulfilling equilibria is well known among economists: because agents expect some particular state of the system to get realized in the future, that very state emerges as an equilibrium outcome for the economy.

While the early works considered the possibility of finite state Markov sunspot equilibria, in the business cycles literature a different class of sunspots is more frequently considered, one in which the sunspot variable has an ARMA form. Examples are found in McCallum (1983) and Farmer (1993).

As Evans and McGough (2011) recently remarked, the fact that sunspot equilibria are theoretically possible in a model does not make them necessarily relevant from an economic perspective, as it might not be possible for agents to coordinate on such equilibria. Because of this, a number of authors have tried to understand the conditions under which sunspot equilibria are learnable. Woodford (1990), Evans and Honkapohja (1994a) and Evans and Honkapohja (2003) analyze learnability for finite state Markov sunspot equilibria, while Evans and Honkapohja (1994b) and Evans and McGough (2005a, 2005b) show that also sunspot solutions in ARMA form can be learnable. Evans and McGough (2011) show in a purely forward looking model that when finite state Markov sunspots are stable under learning, all sunspot equilibria are, provided a common factor representation is used.

All these works take a representative agent approach, and consider only the possibility of all agents conditioning their expectations on an extraneous sunspot component. Heterogeneity in expectations, though, has attracted increasing interest in the recent literature, as it is recognized that it represents a real world feature that economists must take into account in their understanding of expectations formation. In particular, the possibility of different predictors being endogenously chosen on the basis of their relative performance has been investigated in different contexts. From the seminal work of Brock and Hommes (1997), a number of works have analyzed the evolutionary selection of forecasting rules and their impact on economic outcomes. Recent examples include Branch and Evans (2006), Hommes (2009), Guse (2010) and Berardi (2011).

Much less investigated so far has been the link between heterogeneity and sunspot equilibria. A notable exception is Berardi (2009), who shows the possibility of heterogeneous equilibria, where only a fraction of agents use a sunspot variable in their forecasts, to emerge in a purely forward looking model, but who also points out the fragility of such equilibria under predictor choice dynamics. If agents are allowed to choose endogenously whether to include or not a sunspot in their forecasting model, based on a mean squared error measure of performance, it does not exist an equilibrium where only a fraction of agents uses the sunspot.

The aim of this work is to build on Berardi (2009) and extend the analysis to a more general framework that includes lagged endogenous variables. We investigate if the stable sunspot solutions found by Evans and McGough (2005a) under adaptive learning are also stable under heterogeneity and endogenous selection of forecasting models. An important result here will be that some degree of coordination is required among agents in order for the sunspot solution to emerge. Moreover, we analyze the possibility of having heterogeneous solutions where i) only a fraction of agents uses a sunspot variable; and where ii) different groups of agents use different sunspots.

We will model the endogenous selection of forecasting rules by agents using replicator dynamics, which represents the evolution of the fraction of agents using each of the possible predictors available. The concept of replicator dynamics is popular in game theory and it is used to model evolutionary dynamics of strategies in the population of players. While it is borrowed from biology, where it was first introduced by Taylor and Jonker (1978) to formalize the notion of evolutionarily stable strategy, Borgers and Sarin (1997) give it a learning interpretation at the individual level. Fudenberg and Levine (1998) provide an extensive treatment in game theory, while Sethi and Franke (1995), Branch and McGough (2008) and Guse (2010) have applied it to macroeconomic settings.

As for the adaptive learning of parameters within each forecasting model, we follow a growing literature in macroeconomics and assume that agents act as econometrician and recurrently estimate those parameters using techniques such as recursive least squares. We make use of the E-stability principle (for a detailed treatment of the concepts and techniques used, see Evans and Honkapohja, 2001) that links stability of an adaptive learning algorithm of this type to the concept of E-stability, which depends on an associated system of differential equations that are much easier to analyze than the stochastic dynamics in real time of the adaptive algorithm. We will therefore use the terms learnable and E-stable interchangeably in this work, where it is understood that the conditions that make these two concepts equivalent are satisfied.

The plan of the paper is as follows: in Section 2 we introduce the model; in Section 3 we analyze the possibility of heterogeneous equilibria with only a fraction of agents using a sunspot in their forecasts, and derive implications for the possibility of sunspot equilibria emerging endogenously in an economy; in Section 4 we investigate the possibility of different groups of agents using different sunspots, considering both the case of uncorrelated and correlated sunspots; Section 5 concludes.

2 The model

We consider the univariate forward looking model

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t, \tag{1}$$

where the exogenous shock v_t is white noise. Under rational expectations, with $E^* = E$, the expectational operator, the model has one unique non-explosive solution if $0 <\mid \phi_1 \mid< 1 <\mid \phi_2 \mid$, with ϕ_1 and ϕ_2 representing the two roots of the polynomial $\beta\phi^2 - \phi + \delta$:

$$\phi_{1,2} = \frac{1 \pm \sqrt{1 - 4\beta\delta}}{2\beta}$$

with ϕ_1 being the root obtained with the $-$ sign. In this case the unique solution takes the form[1]

$$y_t = \phi_1 y_{t-1} + (\beta\phi_2)^{-1} v_t. \tag{2}$$

We will be interested here instead in the case with $0 <| \phi_1 |<| \phi_2 |< 1$, as in this case solutions other than the minimum state variable (MSV) one exist,[2] and sunspots can play a role in the model. To fix ideas, consider the general solution to model (1), that can be written as

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1}\delta y_{t-2} - \beta^{-1} v_{t-1} + \varepsilon_t \tag{3}$$

where

$$\varepsilon_{t+1} = y_{t+1} - E_t y_{t+1}$$

is a martingale difference sequence (mds). By defining

$$\varepsilon_t = (\beta\phi_i)^{-1} v_t,$$

$i \in \{1, 2\}$, we get the two MSV solutions, while if instead

$$\varepsilon_t = (\beta\phi_i)^{-1} v_t + (1 - \phi_i L)\,\xi_t,$$

we have a sunspot solution, where ξ_t is the sunspot component.[3] This last equation imposes a restriction on the sunspot, which must "resonate"[4] with the structural parameters for the economy, i.e., it must have an AR(1) form with coefficient equal to ϕ_i.

Formally, for a given sunspot variable

$$\xi_t = \lambda\xi_{t-1} + \varepsilon_t$$

we must have that

$$\lambda = \phi_i$$

in order for the sunspot to enter into the solution of (1).

It is well known that, for a given solution, we can have alternative representations, as it has been shown by Evans and McGough (2005a, 2005b). Only sunspot solutions with the so called common factor (CF) representation, though, turn out to be learnable. We will thus focus on these solutions in our analysis, and in particular on the only one being learnable:

$$y_t = \phi_1 y_{t-1} + (\beta\phi_2)^{-1} v_t + \kappa\xi_t, \tag{4}$$

with

$$\xi_t = \phi_2\xi_{t-1} + \varepsilon_t.$$

[1] We will restrict our analysis to the case where these roots are real, which imposes the restriction $\beta\delta < 1/4$.

[2] The term "minimum state variables" was introduced by McCallum (1983) and refers to the solution with the minimum possible number of state variables.

[3] For a derivation of the MSV and sunspot solutions from (3), see the Appendix.

[4] The terminology comes from Evans and McGough (2005a).

Existence and E-stability of this solution requires $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.[5] In this case, moreover, the parameter κ attached to the sunspot is free. Note that these conditions imply $\phi_{1,2} < 0$.

An additional result from Evans and McGough (2005a) is that the sunspot-free MSV solution is strongly E-stable when agents parameterized their forecasting model allowing for a sunspot if i) $\beta < 1/2$ or $\beta + \delta < 1$ and (ii) $\beta(\phi_1 + \lambda) < 1$.

3 Heterogeneous expectations

Starting from the results of Evans and McGough (2005a), outlined in the previous section, we want now to introduce heterogeneity of beliefs in this framework and investigate whether the sunspot solutions that have been found to be learnable under adaptive learning dynamics are robust to agents doubting about the relevance of the sunspot in their forecasting model. We therefore allow agents to use one of two different models, one that includes and one that does not include the sunspot component, and choose between the two on the basis of the relative performance in forecasting, as expressed by the expected mean square errors (MSE). In specific, we will use replicator dynamics to model the evolution of the fraction of agents using each model.

The economy is still represented by (1). There is a continuum of agents on the unit interval, and in forming their expectations they can use one of two models or perceived laws of motion (PLM), one sunspot-free (PLM^1)

$$y_t = a_1 + b_1 y_{t-1} + c_1 v_t \tag{5}$$

and one that includes a sunspot (PLM^2)

$$y_t = a_2 + b_2 y_{t-1} + c_2 v_t + d_2 \xi_t. \tag{6}$$

We denote by μ the fraction of agents using the sunspot-free model, and $(1 - \mu)$ the remaining agents using the model with sunspot. Aggregate expectations are therefore given by

$$E_t^* y_{t+1} = \mu E_t^1 y_{t+1} + (1 - \mu) E_t^2 y_{t+1} \tag{7}$$

where $E_t^i y_{t+1}$ are expectations formed using PLM^i, $i \in \{1, 2\}$.

The parameter μ will be regarded as endogenous, and it will be determined by replicator dynamics based on the relative performance of the two models, measured by the unconditional mean square error. Using this measure we will be able to derive analytical results throughout the paper, while if we were instead to use a measure of real time performance we would have to resort to numerical simulations. Moreover, as Branch and Evans (2006) point out, such a measure would

[5] These restrictions are obtained by Evans and McGough (2005a) by simultaneously imposing conditions for indeterminacy, E-stability and real solution. Conditions for E-stability alone would be $\beta < 1/2$ or $\beta + \delta < 1$.

not be appropriate in a stochastic framework such as ours. We will thus have

$$\begin{aligned}\dot{\mu} &= \mu\left\{-MSE^1 - \left[\mu(-MSE^1) + (1-\mu)\left(-MSE^2\right)\right]\right\} \\ &= \mu\left(1-\mu\right)\Delta \end{aligned} \tag{8}$$

where $\Delta = MSE^2 - MSE^1$ and

$$MSE^i = E\left(y_t - E^i_{t-1}y_t\right)^2.$$

Clearly, $\Delta = 0$ implies $\dot{\mu} = 0$, as the two models deliver the same performance and there is no incentive for agents to switch from one to the other. Moreover, also $\mu = 0$ and $\mu = 1$ are resting points for the dynamics of μ: once homogeneity is reached, the excluded model is no longer used.

We will be interested, in particular, to see whether an equilibrium for the dynamics of the two groups of agents exists other than the two homogeneous ones, i.e., if there exists a situation where $\dot{\mu} = 0$ but $\mu \notin \{0, 1\}$: this will require $\Delta = 0$.

3.1 Stability under learning

First, we start by taking μ as an exogenous and given parameter, and focus on the analysis of learnability in presence of heterogeneity. In the next Section we will then bring back replicator dynamics into the framework.

One important thing that must be noted is that the presence of heterogeneity changes the resonant frequency condition for the sunspot, as already pointed out in Berardi (2009). To see how this happens in the present context, and to fix ideas about learning, consider agents using (5) and (6) to form expectations. We then have

$$E^1_t y_{t+1} = a_1\left(1+b_1\right) + b_1^2 y_{t-1} + c_1 b_1 v_t \tag{9}$$

$$E^2_t y_{t+1} = a_2\left(1+b_2\right) + b_2^2 y_{t-1} + c_2 b_2 v_t + d_2\left(b_2+\lambda\right)\xi_t, \tag{10}$$

where we have assumed that λ is known to agents: otherwise, since the sunspot component is exogenous and observable, this parameter could be consistently estimated using OLS techniques. Aggregating expectations and substituting into (1), we get the temporary equilibrium, or actual law of motion (ALM) for the economy:

$$\begin{aligned} y_t &= \beta\left[\mu a_1\left(1+b_1\right) + \left(1-\mu\right)a_2\left(1+b_2\right)\right] + \left[\beta\left(\mu b_1^2 + \left(1-\mu\right)b_2^2\right) + \delta\right] y_{t-1} + \\ &\quad + \left[\beta\left(\mu c_1 b_1 + \left(1-\mu\right)c_2 b_2\right) + 1\right] v_t + \beta\left[\left(1-\mu\right)d_2\left(b_2+\lambda\right)\right]\xi_t. \end{aligned} \tag{11}$$

According to the E-stability principle (see Evans and Honkapohja, 2001), maps from parameters of the PLM into those of the ALM provide the ordinary differential equations (ODES) that govern the asymptotic behavior of adaptive learning dynamics:

$$\dot{a}_1 = \beta\left[\mu a_1\left(1+b_1\right)+(1-\mu)\,a_2\left(1+b_2\right)\right]-a_1 \tag{12}$$
$$\dot{a}_2 = \beta\left[\mu a_1\left(1+b_1\right)+(1-\mu)\,a_2\left(1+b_2\right)\right]-a_2 \tag{13}$$
$$\dot{b}_1 = \beta\left(\mu b_1^2+(1-\mu)\,b_2^2\right)+\delta-b_1 \tag{14}$$
$$\dot{b}_2 = \beta\left(\mu b_1^2+(1-\mu)\,b_2^2\right)+\delta-b_2 \tag{15}$$
$$\dot{c}_1 = \beta\left[\mu c_1 b_1+(1-\mu)\,c_2 b_2\right]+1-c_1 \tag{16}$$
$$\dot{c}_2 = \beta\left[\mu c_1 b_1+(1-\mu)\,c_2 b_2\right]+1-c_2 \tag{17}$$
$$\dot{d}_2 = \beta\,(1-\mu)\left(b_2+\lambda\right)d_2-d_2. \tag{18}$$

Fixed points of these ODEs are possible equilibria for the model.

Definition 1 *An heterogeneous expectations equilibrium with sunspot is an endogenous stochastic process y_t, a fundamental shock v_t and an exogenous stochastic process ξ_t, with population fraction $\mu\in(0,1)$ and a set of expectational parameters $\{a_1,b_1,c_1,a_2,b_2,c_2,d_2\}$ such that: i) y_t solves (1) for any t; ii) expectations are given by (7),(9) and (10); iii) expectational parameters are fixed points of the maps (12)-(18).*

Note that we are requiring $\mu\in(0,1)$, i.e., both PLMs must be used in an heterogeneous equilibrium. If instead $\mu\in\{0,1\}$, we then have an homogeneous expectations equilibrium (with or without sunspot, respectively).

Looking at the set of ODEs (12)-(18), we can see that there is only a symmetric solution where a_i, b_i and c_i take the same value for both groups of agents, and specifically

$$\begin{aligned}\bar{a} &:= \bar{a}_1=\bar{a}_2=0\\ \bar{b} &:= \bar{b}_1=\bar{b}_2=\phi_{1,2}\\ \bar{c} &:= \bar{c}_1=\bar{c}_2=\frac{1}{1-\beta\bar{b}}\end{aligned}$$

while d_2 can either take the value of zero, or be free in case $\beta\left[(1-\mu)\left(\bar{b}+\lambda\right)\right]=1$. As pointed out before, in the homogeneous case only the solution obtained with $\bar{b}=\phi_1$ can be learnable, and this requires $\beta<-1/2$, $\beta+\delta<-1$ and $4\beta\delta<1$. It turns out that the same conditions are required for learnability in the heterogeneous case here analyzed and the new parameter μ (for the moment taken as given) that arises under heterogeneity does not affect these conditions. Consider in fact the system of ODEs for b_1 and b_2: the Jacobian governing stability of this system is given by

$$J_b=\begin{bmatrix}2\beta\mu b_1-1 & 2\beta\,(1-\mu)\,b_2\\ 2\beta\mu b_1 & 2\beta\,(1-\mu)\,b_2-1\end{bmatrix}$$

and it has eigenvalues equal to -1 and $2\beta\mu b_1+2\beta\,(1-\mu)\,b_2-1$. These, evaluated at the equilibrium points $\bar{b}=\phi_{1,2}$ show that the solution for ϕ_1 is always stable, while that for ϕ_2 never is.

Turning now to the ODEs for a_1 and a_2, we get that the Jacobian for this system is given by

$$J_a=\begin{bmatrix}\beta\mu\left(1+b_1\right)-1 & \beta\,(1-\mu)\left(1+b_2\right)\\ \beta\mu\left(1+b_1\right) & \beta\,(1-\mu)\left(1+b_2\right)-1\end{bmatrix}$$

whose eigenvalues, evaluated at $\bar{b} = \phi_1$, show that again stability requires either $\beta < 1/2$ or $\beta + \delta < 1$.

From the ODEs for c_1 and c_2 then we obtain the Jacobian

$$J_c = \begin{bmatrix} \beta\mu b_1 - 1 & \beta(1-\mu) b_2 \\ \beta\mu b_1 & \beta(1-\mu) b_2 - 1 \end{bmatrix}$$

whose eigenvalues reveal again that the solution obtained with ϕ_1 is stable. We therefore have that, at this point, once we add conditions for indeterminacy and real solution to the conditions for E-stability required for parameters a_1 and a_2, we obtain the same region for stability of equilibrium as in the homogeneous case.

We turn now to the dynamics for d_2. The corresponding ODE gives the additional condition for learnability:

$$\beta(1-\mu)(\phi_1 + \lambda) < 1. \tag{19}$$

Before discussing this condition, we must consider the resonant frequency condition, and see how it is modified by the presence of heterogeneity. Considering the learnable solution, with $\bar{b} = \phi_1$, parameter d_2 is now free for $\beta(1-\mu)(\phi_1 + \lambda) = 1$, which requires

$$\lambda = \tilde{\lambda} := \frac{1}{\beta(1-\mu)} - \phi_1 \tag{20}$$

or equivalently, expressed as a resonant fraction[6] condition

$$\mu = \tilde{\mu} := 1 - \frac{1}{\beta(\phi_1 + \lambda)}. \tag{21}$$

We see that heterogeneity modifies the condition required for the sunspot to resonate with the economy, and it is no longer the case that the AR(1) coefficient of the sunspot must be the same as ϕ_2, as it was previously the case. Only if $\mu = 0$, i.e., all agents in the economy use the sunspot model, this condition is restored.

The resonant condition under heterogeneity can be interpreted in two ways: for a given fraction of agents using a sunspot variable in their forecasts (i.e., for a given μ), the AR(1) coefficient for the sunspot must be as defined by $\tilde{\lambda}$ in (20) in order to resonate with the economy and generate a sunspot equilibrium; alternatively, for a given exogenous AR(1) stochastic process (i.e., for a given λ), there is a fraction of agents $\tilde{\mu}$, as defined in (21), that resonates with the economy and gives rise to a sunspot equilibrium.

Going back to condition (19) for learnability, note that the resonant frequency/fraction restriction implies that the l.h.s. of condition (19) is equal to one. Following the argument exposed in Evans and Honkapohja (1992) and used also in Evans and McGough (2005a), since $b_2 \to \phi_1$ we have that d_2 converges to a finite value, and therefore no additional stability conditions are required.

Since the belief parameter d is free in equilibrium, there is in fact a continuum of solutions.

[6] This terminology was first used in Berardi (2009).

In this case, since the rest point of the system of ODEs governing E-stability is not an isolated point, the notion of E-stability has to be redefined as in Evans and McGough (2005a) in order to allow for the fact that it must refer to a set of fixed points instead of an isolated one. Moreover, the E-stability principle, which links E-stability and adaptive learning, formally applies only to the case where the system of ODEs has an isolated rest point. Evidence, though, suggests that such a link still holds in this class of models, as pointed out in Evans and McGough (2005a) and Evans and Honkapohja (2001, p. 192). We therefore follow these works and continue to use it as a device to study learnability.

Proposition 2 *Under heterogeneous learning dynamics, there exists an heterogeneous expectations equilibrium with sunspot given by* $\bar{a} = 0, \bar{b} = \phi_1, \bar{c} = \frac{1}{1-\beta\phi_1}$, *and* $\bar{d}$ *free provided* $\lambda = \frac{1}{\beta(1-\mu)} - \phi_1$, *which is E-stable for* $\beta < -1/2$, $\beta + \delta < -1$ *and* $4\beta\delta < 1$.

3.2 Replicator dynamics

We turn allow the relative fraction of agents using each model (μ) to be determined endogenously through replicator dynamics, as specified by equation (8). We start by defining the unconditional expected MSE for the two models as:

$$\begin{aligned} MSE^1 &= E\left(y_t - \theta_1 z_t\right)^2 \\ MSE^2 &= E\left(y_t - \theta_2 z_t\right)^2 \end{aligned}$$

where

$$\begin{aligned} \theta_1 &= [a_1 \ b_1 \ c_1 \ 0] \\ \theta_2 &= [a_2 \ b_2 \ c_2 \ d_2] \end{aligned}$$

and

$$z_t = [1 \ y_{t-1} \ v_t \ \xi_t]'.$$

Using equilibrium values for belief parameters $[a_i,\, b_i,\, c_i]$, and for generic d_2, we have

$$\begin{aligned} MSE^1 &= \left[\beta\left(1-\mu\right)\left(\phi_1+\lambda\right) d_2\right]^2 \sigma_\xi^2 \\ MSE^2 &= \left[\left(\beta\left(1-\mu\right)\left(\phi_1+\lambda\right)-1\right) d_2\right]^2 \sigma_\xi^2. \end{aligned}$$

This means that the replicator dynamics equation (8) can be written as

$$\dot{\mu} = \mu\left(1-\mu\right)\left[1-2\beta\left(1-\mu\right)\left(\phi_1+\lambda\right)\right] d_2^2 \sigma_\xi^2. \tag{22}$$

In order to have $\dot{\mu} = 0$ we need $\Delta = 0$, which, for generic $d_2 \neq 0$, requires $\mu = \hat{\mu}$, where

$$\hat{\mu} = 1 - \frac{1}{2\beta\left(\phi_1+\lambda\right)}. \tag{23}$$

If instead $\mu > \hat{\mu}$, this implies $\Delta > 0$ (and vice-versa).

Definition 3 *An endogenous heterogeneous expectations equilibrium with sunspot is an heterogeneous expectations equilibrium as defined in Definition (1), but where the population fraction μ is endogenously determined by (22).*

Comparing condition (23) with the resonant fraction condition (21) that ensures the existence of the sunspot solution, we can see that the two differ. This means that when the sunspot solution exists (i.e., the resonant fraction condition is satisfied and $\mu = \tilde{\mu}$), the condition for $\Delta = 0$, which would give $\dot{\mu} = 0$, can not be satisfied. In particular, the resonant fraction condition implies $\dot{\mu} < 0$ (since in that case $MSE^2 = 0$), which means that the fraction of agents using the sunspot variable increases at $\tilde{\mu}$.

On the other hand, for a given sunspot process, if the resonant frequency condition is not satisfied ($\lambda \neq \tilde{\lambda}$), could μ adjust so that $\tilde{\lambda} = \lambda$ (note that for a given stochastic process that could represent a sunspot, λ is given and exogenous, so it is $\tilde{\lambda}$ that has to adjust, i.e., $\mu \to \tilde{\mu}$). In other words, is it possible that *any* exogenous variable becomes a sunspot for the model, as μ adjusts and the resonant frequency condition emerges endogenously? To answer this question, we need to check whether $\tilde{\mu}$ is an equilibrium under replicator dynamics: since we know that $\tilde{\mu} < \hat{\mu}$, we have that $\dot{\mu}$ is negative at $\tilde{\mu}$, and this means that $\tilde{\mu}$ can not be an equilibrium for the replicator dynamics. The resonant fraction condition, therefore, can not emerge spontaneously.

Proposition 4 *The resonant fraction condition $\mu = \tilde{\mu}$ required to have an heterogeneous equilibrium under learning dynamics can not emerge endogenously under replicator dynamics.*

We want now to check for stability under replicator dynamics of the two homogeneous solutions, $\mu = 0$ and $\mu = 1$. Note that both imply $\dot{\mu} = 0$, so they are resting points of the population dynamics, but are they locally stable? To answer this question we must check the sign of $d\dot{\mu}/d\mu$ at each of these two points. It turns out that $d\dot{\mu}/d\mu$, evaluated at $\mu = 0$, is negative for $\beta(\phi_1 + \lambda) > \frac{1}{2}$, which substituting for the equilibrium value of ϕ_1 means that under replicator dynamics $\mu = 0$ is stable for $2\beta\lambda > \sqrt{1 - 4\beta\delta}$: in this case the equilibrium with all agents using the sunspot is robust to small deviations. Using the resonant frequency condition when $\mu = 0$, i.e., $\lambda = \phi_2$, this means that replicator dynamics at $\mu = 0$ are always stable when the sunspot solution exists.

What about $\mu = 1$: is the sunspot-free equilibrium stable under a small deviation in the population, or is it enough that few people start using the sunspot for the economy to move away from it? Since the sign of $d\dot{\mu}/d\mu$ at $\mu = 1$ is negative, it turns out that $\mu = 1$ is also always locally stable: if only few people deviate from the fundamental equilibrium and start using the sunspot, they are quickly swept away and the sunspot becomes again irrelevant for the economy.

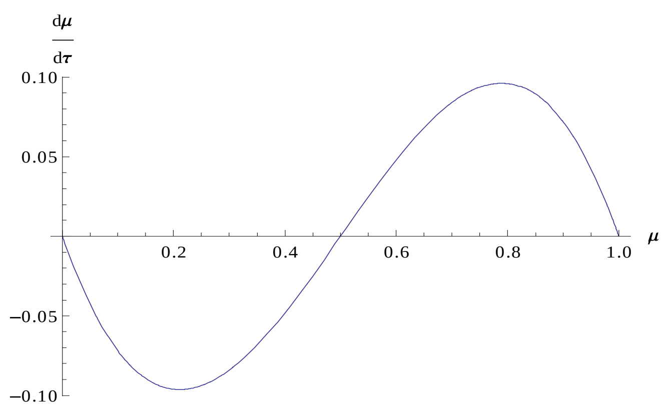
Proposition 5 *The homogeneous equilibrium with $\mu = 1$ is always stable under replicator dynamics. The homogeneous equilibrium with $\mu = 0$ is also stable under replicator dynamics, provided the sunspot solution exists (i.e., the resonant frequency condition is satisfied).*

In addition to the two homogeneous equilibria, we have seen that there is an additional resting point for the replicator dynamics, which obtains for $\mu = \hat{\mu}$: in this case in fact $\Delta = 0$ and therefore $\dot{\mu} = 0$. We have seen above, though, that in this case the resonant fraction condition is not satisfied, so this solution is not an equilibrium in terms of agents' expectations, as their beliefs would be constantly falsified by data. But even so, would it be stable under replicator dynamics? It is possible to show that $d\dot{\mu}/d > 0$ at $\mu = \hat{\mu}$, so the answer is no.

Proposition 6 *The heterogeneous solution with $\mu = \hat{\mu}$ is not stable under replicator dynamics.*

The findings about the two homogeneous solutions lead to a new question: what is the minimum number of people needed to make the sunspot matter in the economy? The answer, it turns out, depends on $\hat{\mu}$. When both $\mu = 0$ and $\mu = 1$ are locally stable, in fact, there must be a threshold in between that separates the two basins of attraction for these equilibria: this threshold is given by $\hat{\mu}$.

In Figure 1 we plot an instance of the behavior of $\dot{\mu}$ that clarifies this point, obtained setting $\beta(\phi_1 + \lambda) = 1$ (which is implied by $\lambda = \phi_2$) and $d_2^2\sigma_\xi^2 = 1$ (the value of $d_2^2\sigma_\xi^2$ is irrelevant for the sign of the derivative, as it is always positive):



In the picture, given the chosen parameters, $\hat{\mu} = 0.5$. As $\beta(\phi_1 + \lambda)$ decreases, $\hat{\mu}$ decreases and ultimately disappears for $\beta(\phi_1 + \lambda) \leq .5$: note, though, that when the resonant frequency/fraction condition is satisfied for the heterogeneous case, $\beta(\phi_1 + \lambda) > 1$ and therefore $\hat{\mu} > 0.5$.

What follows is a detailed explanation of the behavior of the system at the three resting points. At $\mu = 0$, since $d\dot{\mu}/d\mu < 0$, the equilibrium with sunspot is stable. Denoting ε a small fraction of agents deviating from the equilibrium, this means that:

i) at $\mu = 0 - \varepsilon$, $\dot{\mu} > 0$, so $\mu \uparrow 0$ (though this case is of no practical relevance, as $\mu \in [0, 1]$);

ii) at $\mu = 0 + \varepsilon$, $\dot{\mu} < 0$, so $\mu \downarrow 0$: the equilibrium with sunspot is robust to a small fraction of agents switching model and stopping using the sunspot.

At $\mu = 1$, since $d\dot{\mu}/d\mu < 0$, also the sunspot-free equilibrium is stable. This means that

i) at $\mu = 1 - \varepsilon$, $\dot{\mu} > 0$, so $\mu \uparrow 1$: the sunspot-free equilibrium is robust to a small fraction of agents starting using the sunspot in their forecasts;

ii) at $\mu = 1 + \varepsilon$, $\dot{\mu} < 0$, so $\mu \downarrow 0$ (though this case is of no practical relevance, as $\mu \in [0, 1]$).

At $\mu = \hat{\mu}$, since $d\dot{\mu}/d\mu > 0$, the heterogeneous equilibrium is not stable:

i) at $\mu = \hat{\mu} - \varepsilon$, $\dot{\mu} < 0$, so $\mu \downarrow 0$: a small number of additional agents starting using the sunspot is enough to trigger the economy towards the homogeneous sunspot equilibrium where everybody uses the sunspot;

ii) at $\mu = \hat{\mu} + \varepsilon$, $\dot{\mu} > 0$, so $\mu \uparrow 1$: a small number of agents stopping using the sunspot is enough to trigger the economy towards the sunspot free equilibrium, where nobody conditions their forecasts on the sunspot.

These findings imply that we need at least a fraction $(1 - \hat{\mu} - \varepsilon)$ of people using the sunspot for them to take over (and $\mu \downarrow 0$). This means that, especially if $(1 - \hat{\mu})$ is large, in order for the sunspot solution to emerge there must be a lot of coordination among people, as a large fraction of them needs to start using the sunspot variable at the same time.

Proposition 7 *In order for the homogeneous sunspot equilibrium ($\mu = 0$) to emerge, there must be in the economy at least a fraction $(1 - \hat{\mu} - \varepsilon)$ of people that start using the sunspot at the same time.*

This is an important results, as it shows that there must be some degree of coordination among agents for a sunspot to become relevant in an economy. It is not enough, in fact, that a marginal fraction of the population starts using the sunspot for the economy to move to a sunspot equilibrium: this move requires that a significant number of people (up to 50% of the population) simultaneously switch to the sunspot model for their forecasts. The question of how such initial coordination can be reached remains open.

3.3 Adaptive learning and replicator dynamics

We consider now in this section joint learning and replicator dynamics. This means that we need to consider at the same time the ODEs for belief parameters (a_i, b_i, c_i, d_2) that come from adaptive learning and the ODE for population fraction μ given by replicator dynamics. The aim is to understand whether it is possible to have fixed points of the joint dynamics, i.e., to find an equilibrium for the learning and replicator dynamics as they happen simultaneously.

First, we must derive the replicator dynamics equation with out of equilibrium (from a learning perspective) belief parameters. Assuming that both groups of agents start with the same initial beliefs about common parameters (a_i, b_i, c_i), and given that they use the same learning algorithm, we have that even during the learning process, $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. It follows that Δ is given by

$$\Delta = [1 - 2\beta(1-\mu)(b_2 + \lambda)]\, d_2^2 \sigma_\xi^2.$$

Substituting it into the ODE (8) for $\dot{\mu}$, we can see that the only belief parameters that affect the evolution of μ are b_2 and d_2. We can therefore consider the subsystem composed of $\dot{\mu}$, $\dot{b}_2$, $\dot{d}_2$ (plus the E-stability conditions already found for a_i, b_1 and c_i, i.e., $\beta < 1/2$ or $\beta + \delta < 1$). This is given by

$$\begin{aligned}
\dot{\mu} &= \mu(1-\mu)[1 - 2\beta(1-\mu)(b_2 + \lambda)]\, d_2^2 \sigma_\xi^2 \\
\dot{b}_2 &= \beta b_2^2 + \delta - b_2 \\
\dot{d}_2 &= \beta(1-\mu)(b_2 + \lambda)\, d_2 - d_2.
\end{aligned}$$

Its stability is governed by the Jacobian

$$J = \begin{bmatrix} \frac{\delta\dot{\mu}}{\delta\mu} & \frac{\delta\dot{\mu}}{\delta b_2} & \frac{\delta\dot{\mu}}{\delta d_2} \\ \frac{\delta\dot{b}_2}{\delta\mu} & \frac{\delta\dot{b}_2}{\delta b_2} & \frac{\delta\dot{b}_2}{\delta d_2} \\ \frac{\delta\dot{d}_2}{\delta\mu} & \frac{\delta\dot{d}_2}{\delta b_2} & \frac{\delta\dot{d}_2}{\delta d_2} \end{bmatrix} \tag{24}$$

with

$$\begin{aligned}
\frac{\delta\dot{\mu}}{\delta\mu} &= (1-2\mu)\,\Delta + \left[2\beta\left(\mu-\mu^2\right)(b_2+\lambda)\right] d_2^2\sigma_\xi^2,\ \frac{\delta\dot{\mu}}{\delta b_2} = -2\beta\mu\,(1-\mu)^2\, d_2^2\sigma_\xi^2, \\
\frac{\delta\dot{\mu}}{\delta d_2} &= 2\mu\,(1-\mu)\left[1-2\beta\,(1-\mu)\,(b_2+\lambda)\right] d_2\sigma_\xi^2, \\
\frac{\delta\dot{b}_2}{\delta\mu} &= 0,\ \frac{\delta\dot{b}_2}{\delta b_2} = 2\beta b_2 - 1,\ \frac{\delta\dot{b}_2}{\delta d_2} = 0, \\
\frac{\delta\dot{d}_2}{\delta\mu} &= -\beta\,(b_2+\lambda)\,d_2,\ \frac{\delta\dot{d}_2}{\delta b_2} = \beta\,(1-\mu)\,d_2,\ \frac{\delta\dot{d}_2}{\delta d_2} = \beta\,(1-\mu)\,(b_2+\lambda) - 1,
\end{aligned}$$

to be evaluated at $\mu = \{1, 0, 1 - \frac{1}{2\beta(\phi_1+\lambda)}\}$, $b_2 = \phi_1$, $d_2 = free$. The three values for μ correspond respectively to the homogeneous equilibrium with no sunspots, the homogeneous equilibrium where all agents use the sunspot, and the heterogeneous equilibrium where only a fraction ($\hat{\mu}$) of agents uses the sunspot.

For the system to be stable, we need the eigenvalues of this matrix to have all negative real part. Eigenvalues e_i are:

$$\begin{aligned}
e_1 &= \frac{\delta\dot{b}_2}{\delta b_2}, \\
e_{2,3} &= \frac{1}{2}\left(\frac{\delta\dot{\mu}}{\delta\mu} + \frac{\delta\dot{d}_2}{\delta d_2} \pm \sqrt{\left(\frac{\delta\dot{\mu}}{\delta\mu}\right)^2 + \left(\frac{\delta\dot{d}_2}{\delta d_2}\right)^2 - 2\frac{\delta\dot{\mu}}{\delta\mu}\frac{\delta\dot{d}_2}{\delta d_2} + 4\frac{\delta\dot{\mu}}{\delta d_2}\frac{\delta\dot{d}_2}{\delta\mu}}\right).
\end{aligned}$$

It is easy to see that $e_1 = -\sqrt{1-4\beta\delta} < 0$. As for the other two, let's first look at the equilibrium characterized by $\mu = 1$: in this case the remaining two eigenvalues are equal to -1 and $-d_2^2\sigma_\xi^2$: it follows that matrix J is stable and conditions for existence and stability of equilibrium under joint learning and replicator dynamics reduce to those already found before, i.e., $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.

Proposition 8 *Under joint learning and replicator dynamics, the homogeneous sunspot-free equilibrium is locally stable for $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.*

Let's consider now the equilibrium with $\mu = 0$, where all agents use the sunspot variable. In this case the two remaining eigenvalues $e_{2,3}$ of matrix J are $[1 - 2\beta(\phi_1 + \lambda)]\, d_2^2\sigma_\xi^2$ and $\beta(\phi_1+\lambda)-1$. Using the resonant frequency condition for the homogeneous case, $\lambda = \phi_2$, it is easy to verify that the two eigenvalues are equal to $-d_2^2\sigma_\xi^2$ and 0, this last one pertaining to the dynamics of d_2. Again, recalling conditions for indeterminacy, existence of real solution and E-stability for parameters a_i and c_i, we therefore find that stability of equilibrium under joint replicator and learning dynamics require $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$, the same conditions found by Evans and McGough

(2005a) for adaptive learning alone.

Proposition 9 *Under joint learning and replicator dynamics, the homogeneous sunspot equilibrium is locally stable for $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.*

It remains to be considered the heterogeneous sunspot equilibrium with $\mu = \hat{\mu}$. Using results from the previous sections, we know that existence of such an equilibrium under adaptive learning requires $\mu = \tilde{\mu}$, but this relative composition of agents is not an equilibrium for the replicator dynamics, since $\hat{\mu} \neq \tilde{\mu}$. It follows that it does not exist an heterogeneous sunspot equilibrium under joint learning and replicator dynamics.

Proposition 10 *Under joint learning and replicator dynamics, it does not exist an heterogeneous equilibrium with only a fraction of agents using the sunspot variable.*

4 Heterogeneous sunspots equilibria

We have seen that under learning and replicator dynamics, it does not exist an heterogeneous equilibrium where only a fraction of agents uses the sunspot: either everyone uses it, or no one does. We want to investigate now the possibility of having an equilibrium where different agents use different sunspot variables: would it be possible for the economy to be affected by more than one sunspot if different agents were to use them in their forecasts? In particular, we consider two possibilities: first, the case where these different sunspots are independent from each other; and then we introduce a correlation structure between the two variables.

Formally, agents now can use either PLM^1, represented by forecasting model (6), reproduced here for simplicity[7]

$$y_t = a_1 + b_1 y_{t-1} + c_1 v_t + d_1 \xi_{1,t} \tag{25}$$

or the alternative model (PLM^2)

$$y_t = a_2 + b_2 y_{t-1} + c_2 v_t + d_2 \xi_{2,t} \tag{26}$$

with correlation structure between the two sunspots being represented by the symmetric matrix

$$F = \begin{bmatrix} \sigma^2_{\xi_1} & \sigma^2_{\xi_1 \xi_2} \\ \sigma^2_{\xi_1 \xi_2} & \sigma^2_{\xi_2} \end{bmatrix},$$

where $\sigma^2_{\xi_1 \xi_2} \equiv 0$ if the two sunspots are taken to be independent (section 4.1 below) and $\sigma^2_{\xi_1 \xi_2} \neq 0$ otherwise (section 4.2 below). Moreover, the two processes take the form

$$\xi_{1,t} = \lambda_1 \xi_{1,t-1} + \varepsilon_{1,t} \tag{27}$$

$$\xi_{2,t} = \lambda_2 \xi_{2,t-1} + \varepsilon_{2,t}. \tag{28}$$

We are therefore looking for heterogeneous equilibria where both PLMs are used by agents in the economy. We introduce first some definitions to formally capture this idea.

[7] The only difference is the subscript for parameters in the model, which is now 1 becasue it refers to the first model available to agents. We will use subscript 2 for parameters in the alternative PLM.

Definition 11 *An heterogeneous sunspots equilibrium is an endogenous stochastic process* y_t*, a fundamental shock* v_t *and exogenous stochastic processes* $\xi_{1,t}$ *and* $\xi_{2,t}$*, with population fraction* $\mu \in (0,1)$ *and a set of expectational parameters* $\{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\}$ *such that: i)* y_t *solves (1) for any t; ii) expectations are given by (7) using models (25) and (26); iii) expectational parameters are fixed points of the ODEs derived from the learning algorithms.*

Definition 12 *An endogenous heterogeneous sunspots equilibrium is an heterogeneous sunspots equilibrium, as defined in Definition (11), but where the population fraction* μ *is endogenously determined by the replicator dynamics equation (8).*

Again, the population fraction must be $\in (0,1)$ for an heterogeneous equilibrium to exists. If instead $\mu \in \{0,1\}$, we have an *homogeneous sunspot equilibrium* where all agents use the same sunspot component. If this is obtained through replicator dynamics, we call it an *endogenous homogeneous sunspot equilibrium*.

4.1 Uncorrelated sunspots

We consider first the case of uncorrelated sunspots, i.e., the case where the off diagonal elements of F are equal to zero. Assume a fraction μ of agents use model (25), and the remaining faction $(1-\mu)$ uses model (26). The relevant sets of equations for stability of adaptive learning (in addition to those for a_i, b_i and c_i, which remain as before) are then the ODEs

$$\dot{d}_1 = \beta\mu\,(b_1 + \lambda_1)\,d_1 - d_1 \tag{29}$$

$$\dot{d}_2 = \beta\,(1-\mu)\,(b_2 + \lambda_2)\,d_2 - d_2. \tag{30}$$

Equilibrium points are $d_i = 0$ or d_i free if the appropriate resonant frequency/fraction conditions are satisfied. It is evident that we have now two resonant fraction (frequency) conditions, one for each sunspot:

$$\tilde{\lambda}_1 = \frac{1}{\beta\mu} - \phi_1$$

$$\tilde{\lambda}_2 = \frac{1}{\beta\,(1-\mu)} - \phi_1$$

or

$$\tilde{\mu} = \frac{1}{\beta\,(\phi_1 + \lambda_1)} \tag{31}$$

$$(1-\tilde{\mu}) = \frac{1}{\beta\,(\phi_1 + \lambda_2)}. \tag{32}$$

The last two conditions combined imply the restriction

$$\frac{1}{(\phi_1 + \lambda_1)} + \frac{1}{(\phi_1 + \lambda_2)} = \beta. \tag{33}$$

Note that, given the two resonant fraction/frequency conditions, $\delta\dot{d}_1/\delta d_1 = \delta\dot{d}_2/\delta d_2 = 0$, and therefore, using the same argument as before, learning dynamics for these parameters converge. Adaptive learning in this case requires therefore the same conditions we have seen before, those coming from the ODEs for a_i, b_i and c_i, with no additional requirements introduced by the heterogeneity of sunspots used.

Proposition 13 *For a given μ, an heterogeneous sunspots equilibrium with uncorrelated sunspots exists provided conditions (31) and (32) are satisfied. Such an equilibrium is learnable for $\beta < -1/2$, $\beta + \delta < -1$ and $4\beta\delta < 1$.*

In terms of replicator dynamics, we must first derive the MSEs for the two models. Using equilibrium values for all belief parameters, we have that the MSEs for the two groups of agents are now

$$\begin{aligned} MSE^1 &= \left[\beta\left(1-\mu\right)d_2\left(\phi_1+\lambda_2\right)\right]^2\sigma^2_{\xi_2} \\ MSE^2 &= \left[\beta\mu d_1\left(\phi_1+\lambda_1\right)\right]^2\sigma^2_{\xi_1}. \end{aligned}$$

The replicator dynamics are therefore governed by equation (8), where now

$$\Delta = MSE^2 - MSE^1 = \left[\beta\mu d_1\left(\phi_1+\lambda_1\right)\right]^2\sigma^2_{\xi_1} - \left[\beta\left(1-\mu\right)d_2\left(\phi_1+\lambda_2\right)\right]^2\sigma^2_{\xi_2}.$$

There are two homogeneous equilibria, $\mu = 0$ and $\mu = 1$. The first implies $MSE^2 = 0$, while the second implies $MSE^1 = 0$. Moreover, since $d\dot{\mu}/d\mu$ is negative in both case, both equilibria are stable.

In addition to the two homogeneous equilibria, there could be heterogeneous equilibria where both sunspots are used by some agents. This would require $\dot{\mu} = 0$ for $\mu \notin \{0, 1\}$. The condition required is therefore $\Delta = 0$, or $MSE^1 = MSE^2$. For given parameters $\{\beta, \delta, \lambda_1, \lambda_2, \sigma_{\xi_1}, \sigma_{\xi_2}\}$ and fixed $\{d_1, d_2\}$, this obtains when

$$\mu = \hat{\mu} := \frac{d_2\left(\phi_1+\lambda_2\right)\sigma_{\xi_2}}{d_1\left(\phi_1+\lambda_1\right)\sigma_{\xi_1} + d_2\left(\phi_1+\lambda_2\right)\sigma_{\xi_2}}. \tag{34}$$

Combining this condition with conditions (31)-(32) required to have a sunspot solution under learning, we can derive the restriction

$$\frac{d_1}{d_2}\frac{\sigma_{\xi_1}}{\sigma_{\xi_2}} = 1, \tag{35}$$

which is required for having $\hat{\mu} = \tilde{\mu}$, i.e., an heterogeneous sunspot equilibrium under learning and replicator dynamics. Note that before, with only one group of agents using the sunspot variable in their forecasting model, it was not possible to have an heterogeneous equilibrium that contemporaneously satisfied both conditions for learning and replicator dynamics. We have just proved instead that an heterogeneous equilibrium, under both learning and replicator dynamics, is possible when different groups of agents use different sunspots.

Proposition 14 *With uncorrelated sunspots, an heterogeneous sunspots equilibrium under adaptive learning and replicator dynamics exists provided $\frac{d_1}{d_2}\frac{\sigma_{\xi_1}}{\sigma_{\xi_2}} = 1$.*

We have just shown that an heterogeneous sunspots equilibrium exists. Moreover, from the previous analysis, we know that this equilibrium is stable under adaptive learning dynamics. But is it also stable under replicator dynamics? In order to answer this question, we need to consider the sign of $\frac{d\dot{\mu}}{d\mu}$ evaluated at $\mu = \hat{\mu}$: it turns out that this derivative is always positive, for any value of the parameters, and therefore the heterogeneous equilibrium with $\mu = \hat{\mu}$ is not stable under replicator dynamics. This means that, in this model, heterogeneous sunspot equilibria where different agents use different sunspots in their forecasts can not emerge spontaneously under replicator dynamics.

Moreover, the fact that the two homogeneous sunspots equilibria are stable while the heterogeneous one is not, and that $\mu > \hat{\mu} \Rightarrow \dot{\mu} > 0$ and $\mu < \hat{\mu} \Rightarrow \dot{\mu} < 0$, imply that the two homogeneous solutions are the only possible limiting points for this model, and the two basins of attractions of these points are divided by the heterogeneous solution $\hat{\mu}$. Replicator dynamics will always drive the economy towards an equilibrium where all agents use the same sunspot.

Proposition 15 *With uncorrelated sunspots, there exist two endogenous homogeneous sunspot equilibria where all agents use either one of the two sunspots. These equilibria are stable under replicator dynamics. There exists also an endogenous heterogeneous sunspots equilibrium, where agents use different sunspots, but such equilibrium is not stable under replicator dynamics.*

4.2 Correlated sunspots

We turn now to consider the case of correlated sunspots. In this case agents use different variables as sunspots, but these variables are correlated among them: this means that the off diagonal elements of matrix F are different from zero. In particular, we assume that the two sunspots originate from an imperfectly observable common variable, so that

$$\xi_{1,t} = \eta_t + u_{1,t} \tag{36}$$
$$\xi_{2,t} = \eta_t + u_{2,t} \tag{37}$$

with $u_{1,t}$ and $u_{2,t}$ being two i.i.d. observational noise components and

$$\eta_t = \rho\eta_{t-1} + v_t. \tag{38}$$

In terms of equations (27)-(28), we therefore have that $\lambda_1 = \lambda_2 = \rho$,

$$\varepsilon_{1,t} = v_t + u_{1,t} - \rho u_{1,t-1}$$
$$\varepsilon_{2,t} = v_t + u_{2,t} - \rho u_{2,t-1}$$

and the variance covariance matrix F is composed by

$$\sigma^2_{\xi_1} = \sigma^2_\eta + \sigma^2_{u_1} \tag{39}$$
$$\sigma^2_{\xi_2} = \sigma^2_\eta + \sigma^2_{u_2} \tag{40}$$
$$\sigma^2_{\xi_1\xi_2} = \sigma^2_\eta. \tag{41}$$

With this structure in mind, we can consider now the problem of existence and stability of an heterogeneous sunspots equilibrium where agents use different but correlated sunspots.

Since now the two sunspots are correlated, it is not possible to generate the ODEs governing the learning dynamics for parameters d_i by simply mapping each individual PLM into the ALM, and we need instead to project the ALM onto each PLM separately. Using stochastic approximation techniques, the differential equations governing the evolution of d_1 and d_2 are thus given by

$$\dot{d}_1 = [\beta\mu(\phi_1+\lambda_1)-1]\,d_1 + \beta(1-\mu)(\phi_1+\lambda_2)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_1}^2}d_2 \tag{42}$$

$$\dot{d}_2 = \beta\mu(\phi_1+\lambda_1)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_2}^2}d_1 + [\beta(1-\mu)(\phi_1+\lambda_2)-1]\,d_2, \tag{43}$$

where now it results clear that the correlation among sunspots enriches the dynamics for the belief parameters. This system of ODEs admits the sunspot free solution $d_1 = d_2 = 0$ or alternately a sunspot solution with d_1 or d_2 free provided matrix $(A - I)$ is singular, with I the identity matrix and

$$A = \begin{bmatrix} \beta\mu(\phi_1+\lambda_1) & \beta(1-\mu)(\phi_1+\lambda_2)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_1}^2} \\ \beta\mu(\phi_1+\lambda_1)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_2}^2} & \beta(1-\mu)(\phi_1+\lambda_2) \end{bmatrix}.$$

The singularity condition requires (using the fact that $\lambda_1 = \lambda_2 = \rho$)

$$\mu(1-\mu) = \frac{\beta(\phi_1+\rho)-1}{\beta^2(\phi_1+\rho)^2}\frac{\left(\sigma_\eta^2+\sigma_{u_1}^2\right)\left(\sigma_\eta^2+\sigma_{u_2}^2\right)}{\sigma_\eta^2\sigma_{u_1}^2+\sigma_\eta^2\sigma_{u_2}^2+\sigma_{u_1}^2\sigma_{u_2}^2}. \tag{44}$$

It can be seen that in this case, contrary to the case with uncorrelated sunspots, d_1 and d_2 are related, i.e., there is only one degree of freedom in their choice. In other words, once one of the two is (freely) chosen, the other one is pinned down.[8] This means that introducing a correlation structure in the sunspots restricts the choice of belief parameters across agents. Note in fact that before, with uncorrelated sunspots, no cross restriction between belief parameters was required by adaptive learning, though cross restriction (35) was necessary in order to have an heterogeneous equilibrium under replicator dynamics.

Defining the r.h.s. of (44) by γ, it can be seen that there can now be two distinct values of μ for which the resonant fraction condition is satisfied and that ensure the existence of an heterogeneous sunspots equilibrium under adaptive learning. These values are

$$\tilde{\mu}_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\gamma}, \tag{45}$$

which require $0 < 1 - 4\gamma < 1$. If this restriction is not satisfied, instead, there can be no heterogeneous sunspots equilibria.

Provided condition (45) is satisfied, we ask then if these two heterogeneous sunspots equilibria are stable under learning, i.e., if the system of ODEs (42)-(43) is stable. Stability is governed by the eigenvalues of the Jacobian

$$J = \begin{bmatrix} \frac{\delta\dot{d}_1}{\delta d_1} & \frac{\delta\dot{d}_1}{\delta d_2} \\ \frac{\delta\dot{d}_2}{\delta d_1} & \frac{\delta\dot{d}_2}{\delta d_2} \end{bmatrix},$$

[8] In principle, there is a second possibility: if all elements of $(A - I)$ were equal to zero, then we would have both d_1 and d_2 free, i.e., we would have two degrees of freedom in their choice. But such a possibility can never arise with correlated sunspots (i.e., with $\sigma_{\xi_1\xi_2}^2 \neq 0$), as it can be seen by looking at the elements on the minor diagonal of A.

with

$$\begin{aligned}
\frac{\delta \dot{d}_1}{\delta d_1} &= \beta\mu\left(\phi_1+\rho\right)-1 \\
\frac{\delta \dot{d}_1}{\delta d_2} &= \beta\left(1-\mu\right)\left(\phi_1+\rho\right)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_1}^2} \\
\frac{\delta \dot{d}_2}{\delta d_1} &= \beta\mu\left(\phi_1+\rho\right)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_2}^2} \\
\frac{\delta \dot{d}_2}{\delta d_2} &= \beta\left(1-\mu\right)\left(\phi_1+\rho\right)-1.
\end{aligned}$$

In order to have both eigenvalues negative, we need the trace of J to be negative and the determinant to be positive. This is obtained for

$$0 < \frac{3}{2}+\frac{1}{2}\sqrt{1-4\beta\delta}-\beta\rho \text{ and} \tag{46}$$

$$0 < 1-\beta\left(\phi_1+\rho\right)+\beta^2\mu\left(1-\mu\right)\left(\phi_1+\rho\right)^2\left[1-\frac{\left(\sigma_\eta^2\right)^2}{\left(\sigma_\eta^2\right)^2+\sigma_\eta^2\sigma_{u_1}^2+\sigma_\eta^2\sigma_{u_2}^2+\sigma_{u_1}^2\sigma_{u_2}^2}\right]. \tag{47}$$

Note that (44) implies that the second condition holds with equality, i.e., the determinant of J is zero and therefore one eigenvalue of J is equal to zero. This finding is analogous to the fact that, with only one sunspot, the resonant fraction/frequency condition implies that the dynamics of the ODE for the belief parameter attached to the sunspot are governed by a zero eigenvalue. Again, using the same argument as before, this does not create any problems, as long as condition (46) is satisfied. We therefore have the additional condition for learnability

$$2\beta\rho < 3+\sqrt{1-4\beta\delta}. \tag{48}$$

Proposition 16 *An heterogeneous sunspots equilibrium with correlated sunspots exists provided either $\mu=\tilde{\mu}_1$ or $\mu=\tilde{\mu}_2$, where $\tilde{\mu}_{1,2}$ are defined by (45). Such equilibria are learnable if i) $\beta<-1/2$, $\beta+\delta<-1$ and $4\beta\delta<1$; and ii) condition (48) is satisfied.*

Turning now to analyze replicator dynamics, we want to understand i) conditions for stability of the two homogeneous sunspot equilibria ($\mu=1$ and $\mu=0$), and ii) the possibility of having an endogenous heterogeneous sunspots equilibrium. In particular, we want to check whether the resonant fraction conditions required for learnability, i.e., $\mu=\tilde{\mu}_{1,2}$, can also generate an heterogeneous sunspots equilibrium under replicator dynamics.

First, we must derive the MSEs for the two groups of agents:

$$\begin{aligned}
MSE^1 &= \left[\beta\mu\left(\phi_1+\lambda_1\right)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_2}^2}d_1+\left[\beta\left(1-\mu\right)\left(\phi_1+\lambda_2\right)\right]d_2\right]^2\sigma_{\xi_2}^2 \\
MSE^2 &= \left[\beta\mu\left(\phi_1+\lambda_1\right)d_1+\beta\left(1-\mu\right)\left(\phi_1+\lambda_2\right)\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_{u_1}^2}d_2\right]^2\sigma_{\xi_1}^2.
\end{aligned}$$

Stability of the two homogeneous sunspot solutions ($\mu=1$ and $\mu=0$) require $d\dot{\mu}/d\mu$, evaluated at the two equilibria, to be negative. We have

$$\frac{\delta\dot{\mu}}{\delta\mu} = (1-2\mu)\,\Delta + \left(\mu-\mu^2\right)\frac{\delta\Delta}{\delta\mu}$$

which, evaluated at $\mu = 1$, gives

$$\frac{\delta\dot{\mu}}{\delta\mu}\mid_{\mu=1} = \left[\beta\left(\phi_1+\lambda_1\right)d_1\right]^2\left(\sigma^2_{\xi_1} - \frac{\sigma^2_{\xi_1\xi_2}}{\sigma^2_{\xi_2}}\right),$$

and, evaluated at $\mu = 0$, gives

$$\frac{\delta\dot{\mu}}{\delta\mu}\mid_{\mu=0} = \left[\beta\left(\phi_1+\lambda_2\right)d_2\right]^2\left(\sigma^2_{\xi_2} - \frac{\sigma^2_{\xi_1\xi_2}}{\sigma^2_{\xi_1}}\right).$$

It can easily be seen that both derivatives are negative if and only if

$$\left(\sigma^2_{\xi_1\xi_2}\right)^2 < \sigma^2_{\xi_1}\sigma^2_{\xi_2}.$$

The Cauchy-Schwarz inequality implies that, for generic random variables X and Y,

$$\left[cov(X,Y)\right]^2 \leq var(x)var(Y)$$

with equality holding only in case of X and Y being linearly related. It follows from (36)-(37) that in our case the inequality holds strictly, and the two homogeneous sunspot equilibria are both stable.

In terms of our point ii), an heterogeneous sunspots equilibrium under replicator dynamics would require $MSE^1 = MSE^2$. Using the expressions for the mean squared errors presented above, we have the result

$$\Delta = 0 \Leftrightarrow \mu = \hat{\mu} := \frac{d_2\left(\phi_1+\lambda_2\right)\left(\sigma_{\xi_2}+\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_1}\right)}{d_1\left(\phi_1+\lambda_1\right)\left(\sigma_{\xi_1}-\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_2}\right)+d_2\left(\phi_1+\lambda_2\right)\left(\sigma_{\xi_2}-\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_1}\right)}. \tag{49}$$

Note that with no correlation among sunspots, this condition reduces to (34). Moreover, since we have that $\lambda_1 = \lambda_2 = \rho$, we can rewrite (49) as

$$\hat{\mu} := \frac{d_2\left(\sigma_{\xi_2}+\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_1}\right)}{d_1\left(\sigma_{\xi_1}-\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_2}\right)+d_2\left(\sigma_{\xi_2}-\sigma^2_{\xi_1\xi_2}/\sigma_{\xi_1}\right)}. \tag{50}$$

From (45) and (50), it is possible to derive restrictions necessary to have $\hat{\mu} = \tilde{\mu}_{1,2}$: though the analytical conditions are cumbersome and not very revealing, they impose a cross restriction between belief parameters d_1 and d_2, the variance covariance structure of the economy and fundamental parameters β, δ and ρ.

Proposition 17 *An endogenous heterogeneous sunspots equilibrium with correlated sunspots exists provided* $\hat{\mu} = \tilde{\mu}_{1,2}$.

Would this equilibrium be stable under replicator dynamics? The answer depends on the sign of

$$\frac{\delta \dot{\mu}}{\delta \mu} = (1 - 2\mu)\,\Delta + \left(\mu - \mu^2\right)\frac{\delta \Delta}{\delta \mu}$$

evaluated at $\hat{\mu}$. Since at $\mu = \hat{\mu}$ we have $\Delta = 0$, the sign of this derivative depends only on the sign of $\frac{\delta\Delta}{\delta\mu}$, evaluated at $\hat{\mu}$. It can be shown that

$$\frac{\delta \Delta}{\delta \mu}\,|_{\mu=\hat{\mu}} > 0 \Leftrightarrow \hat{\mu} d_1^2 \left(\sigma_{\xi_1}^2 - \left(\sigma_{\xi_1 \xi_2}^2\right)^2 / \sigma_{\xi_2}^2\right) + (1 - \hat{\mu})\, d_2^2 \left(\sigma_{\xi_2}^2 - \left(\sigma_{\xi_1 \xi_2}^2\right)^2 / \sigma_{\xi_1}^2\right) > 0,$$

and again, using the Cauchy-Schwarz inequality, it follows that this condition is always satisfied and therefore the equilibrium with $\mu = \hat{\mu}$ is always unstable under replicator dynamics.

Proposition 18 *With correlated sunspots, the two endogenous homogeneous sunspot equilibria are stable under replicator while the endogenous heterogeneous sunspots equilibrium is not.*

5 Conclusions

In this paper we have analyzed the possibility of heterogeneous equilibria with sunspots to arise in a simple univariate forward looking model with predetermined variables under learning and predictor choice dynamics.

In particular, we have shown the existence, under adaptive learning, of heterogeneous equilibria where only a fraction of agents uses a sunspot in their forecasts, and heterogeneous equilibria where different groups of agents use different sunspots. These equilibria, in order to exist, need to satisfy a resonant fraction/frequency condition that relates the autoregressive parameter of the sunspot(s) to the fraction of agents using the sunspot(s). Both the homogeneous and the heterogeneous equilibria can be learnable by agents if specific restrictions on parameters are satisfied.

But heterogeneous equilibria with sunspots turn out to be fragile once agents are allowed to doubt the relevance of the sunspot components and to choose whether or not to include them in their forecasting model on the basis of forecasting performance. While homogeneous sunspot equilibria are in fact stable under replicator dynamics, heterogeneous equilibria where only a fraction of agents uses the sunspot, or where different groups of agents use different sunspots, are not. These results, therefore, cast some shadows on the relevance of heterogeneous equilibria with sunspots in real life, as such equilibria require agents never to doubt about the importance of the sunspot components for their forecasts.

Moreover, we have shown that there is a minimum fraction of the population that must start using a sunspot variable for the homogeneous sunspot equilibrium to come about under replicator dynamics. This important result casts some shadows also on the relevance of homogeneous sunspot equilibria, as it shows that there must be an initial degree of coordination among agents in order for a sunspot equilibrium to emerge.

We hope that the findings of this paper will help economists better understand the set of conditions under which homogeneous and heterogeneous equilibria with sunspots can emerge in their models, and therefore better understand the possible relevance of such equilibria in actual economies.

6 Appendix

We briefly show here how to derive the MSV and sunspot solutions for model (1). Starting from the general form (3), reproduced here for simplicity

$$y_t = \beta^{-1} y_{t-1} - \beta^{-1} \delta y_{t-2} - \beta^{-1} v_{t-1} + \varepsilon_t \tag{51}$$

different solutions can be obtained by appropriately redefining the error term

$$\varepsilon_{t+1} = y_{t+1} - E_t y_{t+1}.$$

1. MSV solutions. By defining
$$\varepsilon_t = (\beta \phi_i)^{-1} v_t,$$
$i \in \{1, 2\}$, and deleting the common factor $(1 - \phi_i L)$, we get the two MSV solutions
$$y_t = \phi_j y_{t-1} + (\beta \phi_i)^{-1} v_t \tag{52}$$
where $i, j \in \{1, 2\}$ and $j \neq i$.
2. Sunspot solutions. By defining
$$\varepsilon_t = (\beta \phi_i)^{-1} v_t + (1 - \phi_i L)\, \xi_t,$$
and again deleting the common factor $(1 - \phi_i L)$, we have the sunspot solutions
$$y_t = \phi_j y_{t-1} + (\beta \phi_i)^{-1} v_t + \xi_t, \tag{53}$$
where ξ_t is the sunspot component and again $i, j \in \{1, 2\}$, $j \neq i$.

Solution representations (52) and (53) are called "common factor representation", as they are obtained from the general solution (51) by deleting a common factor component.

References

[1] Azariadis, C., 1981. Self-fulfilling prophecies. Journal of Economic Theory 25, 380-396.

[2] Berardi, M., 2009. Heterogeneous expectations, sunspot equilibria and their fragility. Economics Letters 105, 276-279.

[3] Berardi, M., 2011. Fundamentalists vs. chartists: Learning and predictor choice dynamics. Journal of Economic Dynamics and Control 35, 776-792.

[4] Borgers, T., Sarin, R., 1997. Learning through reinforcement and replicator dynamics. Journal of Economic Theory 77, 1-14.

[5] Branch, W., Evans, G.W., 2006. Intrinsic heterogeneity in expectation formation. Journal of Economic Theory 127, 264-295.

[6] Branch, W., McGough, B., 2008. Replicator dynamics in a cobweb model with rationally heterogeneous expectations. Journal of Economic Behavior and Organization 65, 224-244.

[7] Brock, W.A., Hommes, C., 1997. A rational route to randomness. Econometrica 65, 1059-1096.

[8] Cass, D., Shell, K., 1983. Do sunspots matter? Journal of Political Economy 91, 193-227.

[9] Evans, G.W., Honkapohja, S., 1992. On the robustness of bubbles in linear RE models. International Economic Review 33, 1-14.

[10] Evans, G.W., Honkapohja, S., 1994a. On the local stability of sunspot equilibria under adaptive learning rules. Journal of Economic Theory 64, 142-161.

[11] Evans, G.W., Honkapohja, S., 1994b. Learning, convergence, and stability with multiple rational expectations equilibria. European Economic Review 38, 1071-1098.

[12] Evans, G.W., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press.

[13] Evans, G.W., Honkapohja, S., 2003. Existence of adaptively stable sunspot equilibria near an indeterminate steady state. Journal of Economic Theory 111, 125-134.

[14] Evans, G.W., McGough, B., 2011. Representations and sunspot stability. Macroeconomic Dynamics 15, 80-92.

[15] Evans, G.W., McGough, B., 2005a. Stable sunspot solutions in models with predetermined variables. Journal of Economic Dynamics and Control 29, 601-625.

[16] Evans, G.W., McGough, B., 2005b. Monetary policy, indeterminacy and learning. Journal of Economic Dynamics and Control 29, 1809-1840.

[17] Farmer, R.E., 1993. The Macroeconomics of Self-Fulfilling Prophecies. MIT Press, Cambridge, MA.

[18] Fudenberg, D., Levine, D., 1998. The Theory of Learning in Games. MIT Press, Cambridge, MA.

[19] Guse, E.A., 2010. Heterogeneous expectations, adaptive learning, and evolutionary dynamics. Journal of Economic Behavior and Organization 74, 42–57.

[20] Hommes, C.H., 2009. Bounded rationality and learning in complex markets. In: Barkeley Rosser, Jr., J., Handbook of Research on Complexity. Cheltenham, Edward Elgar, pp. 87–123.

[21] McCallum, B.T., 1983. On non-uniqueness in rational expectations models: An attempt at perspective. Journal of Monetary Economics 11, 139-168.

[22] Sethi, R., Franke, R., 1995. Behavioral heterogeneity under evolutionary pressure: macroeconomic implications of costly optimisation. Economic Journal 105, 583-600.

[23] Taylor, P.D., Jonker, L.B., 1978. Evolutionarily stable strategies and game dynamics. Mathematical Biosciences 40, 145-156.

[24] Woodford, M., 1990. Learning to believe in sunspots. Econometrica 58, 277-307.