The maximum debt-GDP ratio and endogenous growth in the Diamond overlapping generations model:
Three overlapping generations are better than two

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Abstract:

While the public debt has an interior maximum in the Diamond OLG model, due to an inherent nonlinearity [Rankin and Roffia (2003)], this feature also extends to a linear, AK model when it is conjoined with a backward-looking adjustment process for public debt [Braeuninger (2005)]. We show that if the debt dynamics are forward-looking, the maximum will instead be at a *degeneracy* – another possibility considered by Rankin and Roffia. However, the main point of the present paper is that any debt maximum in a finite-horizon model will be of an implausibly low order of magnitude, unless households save over at least two periods. This is because the debt *flow* crowds-out investment, while this is synonymous with the debt s*tock* in a model with only two, non-altruistic, overlapping generations, thus leading to a low maximum stock by default. Removing this restriction produces plausible results, and allows a low rate of economic growth to be a cause as well as a consequence of a high public debt.

1. Introduction

Fiscal sustainability has traditionally focussed on the deficit rather than on the debt with an emphasis on the stability rather the existence of an equilibrium with public debt. See, for example, papers by Nielsen (1992), Bohn (1995) and Chalk (2000). A main question has been the convergence of public deficits or surpluses and thence of the stock of debt. From a forward-looking, Ricardian perspective, stability would require that budget surpluses in present value terms decline asymptotically, so that expenditures are under control, that taxes are forthcoming and that net flows are discounted by a larger factor than the one at which they grow. In an infinite-horizon model or from the perspective of an altruistic and dynastically-minded individual, who is indifferent to the time-profile of taxation, the size of any convergent stock of debt does not matter [Barro (1974)] in the absence of other distortions, and thus the notion of a maximum debt is meaningless.

More recently, Rankin and Roffia (2003) analysed public debt in the very different setting of the Diamond (1965) overlapping generations model where non-altruistic households live finite lives. In this model, where debt clearly does matter, even as one that was originally formulated to analyse this very issue, they found that it has a maximum value because of the nonlinearity of the model, namely, that the capital stock is an increasing and concave function of its own past value through the wage-saving relationship. They designate this as a *bifurcation*, because, from a reverse perspective, there are two steady states that would join up at an interior point where the debt is at a maximum. These authors also point out the alternative possibility of a corner or *degenerate* maximum, where the maximum debt eradicates the economy, which arises as a result in Rankin's (2012) application of the analysis to the Blanchard-Yaari model of perpetual youth [Yaari (1965) and Blanchard (1985)].

There have been a number of offshoots from the original paper. Braeuninger (2005) applies the analysis to a model of growth. Farmer and Zotti (2010) also obtain similar results in an open-economy extension. Roberts (2013) returns to the earlier closedeconomy form to look at flexible-price government debt to investigate the effect of different fiscal policy rules.[1]

However, the paper most relevant to our present concern is Braeuninger (2005). Bifurcation maxima naturally depend on the nonlinearity model and would, *prima facie*, be precluded from appearing in the linear class of AK, endogenous growth models, as expounded by Romer (1986) and exemplified by Lucas (1988). Braeuninger, however, shows that the combination of a standard AK model with a familiar, backward-looking dynamic process for the public debt generates two steady state solutions on either side of another bifurcation point at which there is a maximum sustainable debt-GDP ratio and a corresponding minimum rate of economic growth.

We extend Braeuninger's analysis in two directions by first considering an alternative form of debt dynamics and then by generalising the underlying OLG model. The dynamic stability condition for a backward-looking debt is for the rate of economic growth to exceed the rate of interest. While this condition has merit and also some empirical relevance, we also consider debt from an alternative, Ricardian, forward-looking perspective, as the present value of future budget surpluses. This entails a reversal of the dynamic stability condition, as the present becomes a function of the expected future instead of the actual past.

After initially confirming the bifurcation result of Braeuninger (2005) for a backward-looking debt, we find that if the debt is forward-looking, the maximum is instead at a point of degeneracy. The reason is that the new *forward-stability* condition implies a positive rather than a negative steady state relationship between debt and growth in. Numerical values are applied to flesh-out the model to give an idea of possible magnitudes. It makes no sense to make a comparison of the two cases, because they are predicated on the different parameter assumptions that are needed for the separate

[1] These are found to be important both for the nature and for the size of the maximum debt. A bifurcation maximum also occurs where tax revenue instead of the level of the debt is treated as exogenous, but at higher level

stability conditions. However, a key and unambiguous result is that the computed maximum values for each of the two dynamic cases are inordinately low – even after the OLG measures of the debt-GDP ratio have been converted into more familiar and, necessarily, larger annual figures.

This prompts the second generalisation of the model to three overlapping generations, which allows households to save in more than a single period.   We then obtain plausibly high values for the debt-GDP ratio by making this quite a minor generalisation of the Diamond OLG model. The standard form of the model imposes by default a synonymity of stocks and flows, which is not generally problematic except when there is a specific concern for stocks. The problem is that the investment crowding-out is caused by *flows* of public debt, which places limits on the size of the *stock* of public debt when the two are effectively synonymous.   A corollary is that when economic growth is absent, so that stock changes net out to zero, asset crowding-out is precluded altogether, and the size of the maximum debt depends only on the contractionary effect of debt-servicing taxes, which are unavoidably present if debt represents future surpluses, but may be hypothetically absent if debt constitutes the accumulation of past deficits.   In this event, the size of a backward-looking debt may be extremely high indeed – at least in the steady state.

The paper is organised as follows. In *Section 2*, we present the standard form of the model with two overlapping generations. This is used for the analysis of backward- and forward-looking debt in the next two *Sections*, *3* and *4*. *Section 5* then generalises the model to one of three overlapping generations, allowing saving in two periods. Then, *Sections 6* and *7* can revisit and redraw the analysis of *Sections 3* and *4*.   The concluding *Section 8* provides a brief summary.

2. The basic 2-OLG model

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of what is now an endogenous debt. By contrast, with exogenous income tax rates, rendering both tax revenue and the debt endogenous, the maximum is at point of degeneracy, provided that the first *Inada condition* holds.

2.1 *Production*

There is a Cobb-Douglas production function with internal constant returns to scale in capital and labour, as well in in the capital stock, both internal and external to the firm. Thus, it may thus be presented in per capita terms,

$$y_t(i) = Ak_t^{1-\alpha} k_t(i)^{\alpha} \tag{1}$$

The wage and interest factor facing the firm are $w_t(i)$ and $R_{K.t}$. Marginal cost pricing implies $w_t(i) = (1-\alpha)Ak_t^{1-\alpha} k_t(i)^{\alpha}$ and $R_{K,t} = \alpha Ak_t^{1-\alpha} k_t(i)^{\alpha-1}$, and symmetric equilibrium, $k_t(i) = k_t$, $\forall i$, gives

$$y_t = Ak_t, \quad w_t = (1-\alpha)Ak_t, \quad R_{K,t} = \alpha A. \tag{2}$$

There is a lag between saving and investment, and the assumptions of full depreciation within the period and of no population growth give

$$k_{t+1} = s_t^K \tag{3}$$

Households may save by also holding public debt, d_t,

$$s_t = s_t^K + d_t \tag{4}$$

2.2 *Households*

They also have two period utility functions of the form,

$$U_t = \ln c_t^Y + \lambda \ln c_{t+1}^O, \tag{5.1}$$

where λ is a time-preference factor. Households supply a fixed unit of labour when young for which they receive a wage, w_t, which is taxed at the rate τ. The amount they save accumulates by the gross interest factor of return R, $R \equiv 1 + r \le R_K$, where r is the net-of-tax interest rate. Only the interest rate is taxed and there is no inflationary loss to the principal, so that the net real interest return is $1+(1-\tau)r$. The household budget constraints when young and old are $c_t^Y = (1-\tau)w_t - s_t$ and $c_{t+1}^O = (1+(1-\tau)r)s_t$.

The logarithmic nature of the utility function, whereby the income and substitution effects of the interest rate exactly cancel, plus the absence of a second period earned income to be discounted imply that saving depends only on the take-home wage,

$$s_t = \left(\lambda/(1+\lambda)\right)(1-\tau)w_t \tag{6}$$

Equations (3), (4) and (6) give rise to a further one for capital accumulation,

$$k_{t+1} = \left(\lambda/(1+\lambda)\right)(1-\tau)(1-\alpha)Ak_t - d_t \tag{7}$$

Defining the growth factor and the debt-GDP ratio as

$$G_{t+1} \equiv 1+g \equiv k_{t+1}/k_t\,, \qquad \delta_t \equiv d_t/y_t = d_t/Ak_t\,, \tag{8}$$

enables equation (7) to be presented as

$$G_{t+1} = (1-\tau)G^* - A\delta_t \qquad \text{where} \qquad G^* \equiv \left(\lambda/(1+\lambda)\right)(1-\alpha)A \tag{9.1}$$

The parameter G^*, $G^* > 1$, is defined as benchmark level of the growth factor in the absence of both debt and taxes. A steady state of positive output, of course, requires that the levels of debt and taxes are never so high that $G < 1$ (or $g < 0$). Thus, a degenerate maximum is defined precisely where $G = 1$ (or $g = 0$).

3. Backward-looking public debt in the 2-OLG model

The government financing requirement is given by

$$d_t = E_t - T_t + \left(1+(1-\tau)r\right)d_{t-1}\,, \tag{10.B}$$

where the size of the debt depends on primary public expenditure, E_t, less the revenue raised from taxing the factors of production, T_t, plus the amount of net-of-tax servicing required to service its previous level, d_{t-1}. This public debt in this sense is “backward-looking” as the accumulation of past primary deficits,

$$d_t = \sum_{i=0}^{\infty}\left(1+(1-\tau)r\right)^{-i}(E_{t-i} - T_{t-i}) \tag{11.B}$$

Using additional terms to denote the share of government expenditure and the tax take,

$$\gamma_t \equiv E_t/Y_t\,, \qquad \tau_t \equiv T_t/Y_t\,, \tag{12}$$

allows us to present equation (10.B) in a scale-free, ratio form,

$$\delta_t = \gamma_t - \tau_t + \left(\frac{1+(1-\tau)r}{G_t} \right) \delta_{t-1}. \tag{13.B}$$

A necessary condition for stability is that $G > 1 + (1-\tau)r$, which then ensures convergence to a debt ratio of

$$\delta = \frac{G}{G - \left(1 - (1-\tau)r\right)} (\gamma - \tau) \tag{14.B}$$

This necessary *backward-stability* condition, $G > 1 + (1-\tau)r$ and the condition of a positive debt, $\delta > 0$, requires a steady state of primary *deficits (* $\gamma > \tau$).

It is possible to question the general empirical validity of this stability condition outside a financial repression case of low interest rates.[2] Defining *financial market efficiency* as a case where savers receive the full return on capital, $R = R_K$, implies that the necessary stability condition, according to equations (2) and (9.1), is $\left(\lambda/(1+\lambda)\right)(1-\alpha) > \alpha$. Taking the capital share to have a stylised value of one-third, $\alpha = 1/3$, means that $\lambda > 1$ is required, which is to say that the household is so patient as to value future consumption more than current consumption. Thus, the case of financial repression where savers fail to obtain the marginal product of capital, $R < R_K = \alpha A$, supports the present case. This need not be a pathological case, but merely one where financial market concentration leads monopsonistic rates of return[3] to which the return on public debt is arbitraged accordingly.[4]

Another possibility is that there are competitive financial markets, but a fiscal policy of positive (negative) feedback from the debt to the primary surplus (deficit), $\gamma_t - \tau_t = f(\delta_{t-1})$, $f'(\delta_{t-1}) < 0$. This relationship was found by Bohn (1998) for the

[2] The dynamic efficiency condition for an economy with *exogenous* growth, which is often assumed to hold, is the reverse of this condition.
[3] There is an explicit consideration of this in Roberts (2014).
[4] Bond yields are also typically lower in generally being safer. The factor of relative risk is also worth exploring.

US economy and confirmed by Greiner, Koeller and Semmler (2007) for selected Eurozone countries, obviously reflecting the fact that actual economies are away from their steady states. Linearizing around a steady-state, the feedback rule,

$f(\delta_{t-1}) \approx f(\delta) + f'(\delta)(\delta_{t-1} - \delta)$, then gives

$$\delta_t = (\gamma - \tau - f'(\delta)\delta) + \left(\frac{1+(1-\tau)r}{G_t} + f'(\delta) \right)\delta_{t-1}$$

While this to some extent begs the stability question to assume that primary deficits may be adjusted sufficiently in all eventualities, in terms of the model it may largely be regarded as a redefinition of the terms in equation (13B).

The main issue that is critical to the analysis is the endogeneity of economic growth, which according to equation (9.1), requires that equation (13.B) be amended to

$$\delta_t = \gamma_t - \tau_t + \left(\frac{1+(1-\tau)r}{(1-\tau)G^* - A\delta_{t-1}} \right)\delta_{t-1}, \qquad \frac{\partial \delta_t}{\partial \delta_{t-1}} = \frac{(1+(1-\tau)r)(1-\tau)G^*}{((1-\tau)G^* - A\delta_{t-1})^2} > 0,$$

$$\frac{\partial^2 \delta_t}{\partial {\delta_{t-1}}^2} = \frac{2(1+(1-\tau)r)(1-\tau)G^* A}{((1-\tau)G^* - A\delta_{t-1})^3} > 0 \qquad (15.B1)$$

The debt-ratio is now characterised as a non-linear, first-order difference equation, as its dual for economic growth,

$$G_{t+1} = ((1-\tau)G^* + 1 + (1-\tau)r - A(\gamma - \tau)) - \frac{(1+(1-\tau)r)(1-\tau)G^*}{G_t}$$ [5]

$$\frac{\partial G_{t+1}}{\partial G_t} = \frac{(1+(1-\tau)r)(1-\tau)G^*}{{G_t}^2} > 0, \quad \frac{\partial^2 G_{t+1}}{\partial {G_t}^2} = \frac{2(1+(1-\tau)r)(1-\tau)G^*}{{G_t}^3} < 0 \qquad (16.B1)$$

If an equilibrium exists, the generalised condition for local stability, $(1+(1-\tau)r)(1-\tau)G^* < G^2$, is most likely to hold for a negligible debt, $\delta \approx 0$, where $\gamma \approx \tau$, in which case $1+(1-\tau)r < (1-\tau)G^*$. An even more favourable

[5] A backward-looking debt is clearly a source of a monotonic, adjustment process for economic growth.

case is where $\tau \approx 0$, which leads to $R < G^*$ – or $r < g^*$, which may be regarded as a necessary but not sufficient condition for a backwardly stable debt.

Lemma One: There are dual equilibria for growth and for public debt, if the debt dynamics are backward-looking.

Substituting (15b) into (9) gives the implicit function,

$$F \equiv G-(1-\tau)G^* + \frac{AG}{G-\left(1+(1-\tau)r\right)}(\gamma-\tau)=0, \text{ where}$$

$$\frac{\partial(\gamma-\tau)}{\partial G} = -\frac{1-\left(A/\left(G-\left(1+(1-\tau)r\right)\right)^2\right)\left(1+(1-\tau)r\right)(\gamma-\tau)}{AG/\left(G-\left(1+(1-\tau)r\right)\right)} \quad . \tag{17.B1}$$

The derivative implies a non-monotonic relationship between growth and the deficit, $\gamma-\tau$, if the latter is positive, $\gamma-\tau>0$, which translates into a non-monotonic relationship between growth and the debt ratio, because of the positive relationship between the debt and the deficit,

$$\frac{\partial\delta}{\partial(\gamma-\tau)} = \frac{G}{G-\left(1+(1-\tau)r\right)}\left(1-\frac{A(\gamma-\tau)\left((1+(1-\tau)r\right)}{\left(G-\left(1+(1-\tau)r\right)\right)^2}\right)^{-1} > 0$$

Thus, there is the possibility of two quadratic steady state solutions for growth,

$$G_1, G_2 = \frac{(1-\tau)G^* + \left(1+(1-\tau)r\right) - A(\gamma-\tau)}{2} \pm \sqrt{(..)^2 + \left(1+(1-\tau)r\right)(1-\tau)G^*} \tag{18.B1}$$

as well as for the debt-GDP ratio,

$$\delta_1, \delta_2 = \frac{(1-\tau)G^* - \left(1+(1-\tau)r\right) + A(\gamma-\tau)}{2A} \pm \sqrt{(..)^2 + \frac{(1-\tau)G^*(\gamma-\tau)}{A}} \tag{19.B1}$$

We consider three possibilities in the following *Proposition*.

Proposition One: In the 2-OLG model with backward-looking public debt, there is a critical maximum public expenditure ratio $\gamma = \tilde{\gamma}$ given by $\tilde{\gamma} - \tau \equiv A^{-1}\left((1-(1-\tau)r) + (1-\tau)G^* - 2\sqrt{(1+(1-\tau)r)(1-\tau)G^*}\right) > 0$, which according to the backward-stability condition, $1+(1-\tau)r < (1-\tau)G^*$, is strictly positive in value.

(1) If $\gamma = \gamma$, there is a unique solution for economic growth, $\tilde{G} = G_1 = G_2 = \sqrt{(1+(1-\tau)r)(1-\tau)G^*}$, and for the debt ratio, $\tilde{\delta} = \delta_1 = \delta_2 = A^{-1}\left((1-\tau)G^* - \sqrt{(1+(1-\tau)r)(1-\tau)G^*}\right)$.[6]
This single steady state is borderline stable as $\partial G_t / \partial G_{t-1}\big|_{\tilde{G}} = 1$ and $\partial \delta_t / \partial \delta_{t-1}\big|_{\tilde{\delta}} = 1$.

(2) If $\gamma < \tilde{\gamma}$, there are two steady states, $\delta_1 < \tilde{\delta} < \delta_2$ and $G_1 < \tilde{G} < G_2$. The monotonicity and convexity of the debt adjustment process implies that the lower valued debt steady state is locally stable: $\partial \delta_t / \partial \delta_{t-1}\big|_{\delta_t = \delta_{t-1} = \delta_1} < 1$ and $\partial \delta_t / \partial \delta_{t-1}\big|_{\delta_t = \delta_{t-1} = \delta_2} > 1$, while, correspondingly, the higher valued growth outcome is too, $\partial G_t / \partial G_{t-1}\big|_{G_t = G_{t-1} = G_1} > 1$ and $\partial G_t / \partial G_{t-1}\big|_{G_t = G_{t-1} = G_2} < 1$. A *correspondence principle* applies, since only the stable equilibrium has the regular comparative static properties, $\partial \delta_1 / \delta(\gamma - \tau) > 0$ and $\partial G_1 / \delta(\gamma - \tau) < 0$, while the comparative static properties of the unstable steady state are perverse in having reversed signs.

(3) If $\gamma > \tilde{\gamma}$, no steady state exists, and the adjustment properties of the model point to exploding debt and to imploding growth.

3.2 The solution for the maximum debt

[6] The bifurcation value of $\tilde{\gamma}$ can be solved from either the debt or the growth equation, because the latter is a linear function of the former through equation (9).

The first case $\gamma = \tilde{\gamma}$ implies the maximum steady state for the debt ratio, $\delta = A^{-1}\left((1-\tau)G^* - \sqrt{(1+(1-\tau)r)(1-\tau)G^*}\right)$ and a corresponding, minimum for the steady state growth factor, $G = \sqrt{((1-\tau)G^*)(1+(1-\tau)r)}$. The debt is clearly highest where taxes are at their lowest, because this raises the values both of the primary deficit, and of economic growth. If $\tau = 0$, the respective values are $\delta = A^{-1}\left(G^* - \sqrt{RG^*}\right)$ and $G = \sqrt{G^* R}$. The bifurcation occurs where the growth factor is an unweighted, geometric average of the debt/tax-free growth factor and the interest factor.[7] We note that the requirement $G \geq 1$, places limits on the extent of any financial repression in $R \geq 1/G^*$.

3.3 A valuation of the steady state maximum when debt is backward-looking

We consider possible parameter values. Throughout the analysis we assume that the debt/tax-free annual growth rate is 2.5%, but that $\tau = 0$ is set in order to obtain the largest maximum value for this backward-looking case. For this two period version of the model, the OLG periods are assumed to last 36 years, which means that $G^* = 2.4325$. We then consider three possibilities for the annualised interest rate at which households save, 0.25%, 1.25% and 2.25%. These choices suffice to give the bifurcation values for G. Then after setting $\lambda = 1$ and $\alpha = 1/3$, the value for A, which is consistent, is pinned down, which then leads to the generation of further values for the maximum debt ratio, δ, and for the resulting public expenditure ratio, γ.

3.4 An annualised measure of the debt-GDP ratio

It is also of some interest to obtain more familiar annualised measures of the debt-GDP ratio rather than those based on an income stream spanning the half-life of a drawn-out OLG period. We propose the following procedure for converting the OLG debt ratio

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[7] Note that the requirement $G > 1$ limits the extent of possible in financial repression to $R > 1/G^*$.

into an annualised measure. By definition, $\delta \equiv d/y = (d/Gk)G(k/y)$. On the basis of equations (3) and (4), the term d/Gk constitutes the amount of saving that goes into debt relative to capital/deposits. This in principle should not depend on the length of the period, thus as an analytical step, we can set this portfolio ratio at an arbitrary ρ, $d/Gk = \rho$. The Cobb-Douglas constant factor income shares property also implies that $R_K k = \alpha y$, where α is another constant. Thus, the debt-ratio is $\delta = \alpha\rho G/R_K$, whether in terms of an OLG period, $\delta^{OLG} = \alpha\rho G^{OLG}/R_K^{OLG}$ or of single years, $\delta^{pa} = \alpha\rho G^{pa}/R_K^{pa}$. It then follows that

$\delta^{pa} = \left((R^{OLG}/R^{pa})/(G^{OLG}/G^{pa})\right)\delta^{OLG}$. This may be evaluated by using the compound relationships, $R^{OLG} = R^{pa^L}$ and $G^{OLG} = G^{pa^L}$, where L is the length of the period in terms of years to obtain

$$\delta^{pa} = \left(G^{OLG}/R^{OLG}\right)^{1/L-1}\delta^{OLG} = \left(R^{pa}/G^{pa}\right)^{L-1}\delta^{OLG} \tag{20}$$

This equation along with the choices of values generate the following *Table*.

Table One: Values at the bifurcation of the maximum for backward-looking debt with two overlapping generations: $g^{pa*} = 2.5\%$

	$r^{pa} = 0.25\%$	$r^{pa} = 1.25\%$	$r^{pa} = 2.25\%$
g^{pa}	1.36%	1.87%	2.37%
γ	6.04%	2.36%	0.1206%
δ^{2OLG}	10.98%	6.61%	1.43%
δ^{pa}	26.06%	15.68%	3.40%

We can only conclude that the model, as it now stands, cannot to deliver plausible values for the maximum debt-GDP ratio, even when the heroic assumption of zero taxes is made to boost them. The best case is where interest rates are at their lowest, giving 26% for the annualised measure, a figure, which might be regarded as small public debt in practice, but one which presages a catastrophic loss of a steady state equilibrium in the model.

4. Forward-looking public debt in the 2-OLG model

4.1 Solution with forward-looking debt dynamics

The analysis so far has in essence been a reworking of Braeuninger (2005). We now how it changes qualitatively, when public debt is forward-looking according to,

$$d_t = -(E_t - T_t) + \left(1 + (1-\tau)r\right)^{-1} d_{t+1} \tag{10.F}$$

The forward solution for debt is solved as the expected present value of future surpluses,

$$d_t = -\sum_{i=0}^{\infty} (1 + (1-\tau)r)^{-i} (E_{t+i} - T_{t+i}) \tag{11.F}$$

This consideration is straightforward for models of infinitely-lived households or dynasties, but less so for the basic form of the OLG model.

We suggest that it might also be applied here, if the debt instrument is specified to compensate for the finiteness of households' planning horizons. This requires that, first, it is constituted by perpetuity bonds that have a flexible market price which reflects the present value of future of coupon payments, Secondly, but more hypothetically, that these coupon payments, instead of being fixed cash amounts, consist of shares of the primary surplus.[8] Thus the relevant debt instrument mimics the characteristics of financial equity, making it difficult to escape the notion that determining the size of the largest possible public debt amounts to a fiscal policy of maximizing transfers to bond-holders at the expense of both other tax-payers and the beneficiaries of primary government expenditure.

Re-using the definitions in (8), gives a debt-GDP ratio of

$$\delta_t = -\frac{G_{t+1}}{1+(1-\tau)r}\left((\gamma_{t+1}-\tau_{t+1}) + \frac{G_{t+2}}{1+(1-\tau)r}(\gamma_{t+2}-\tau_{t+2}) + \frac{G_{t+2}G_{t+3}}{\left(1+(1-\tau)r\right)^2}(\gamma_{t+3}-\tau_{t+3}) + .. \right)$$

The size of the debt now depends on expectations of the trajectories of fiscal policy along with the anticipated effects of economic growth. A solution exists if the infinite sum of present value *primary surpluses* is bounded, which requires that primary surpluses are ultimately discounted by a greater factor than the one by which they grow. Thus there is a reversal of the previous stability condition, since the condition $G < 1+(1-\tau)r$ or $g < (1-\tau)r$ is now required for adding up over an infinite future. *Financial efficiency* is most favourable to this case, defined as the situation where households receive the highest possible return on the saving.

In a steady state, the debt ratio given by

$$\delta = \frac{G}{1+(1-\tau)r - G}(\tau - \gamma). \qquad (14.F)$$

which is notable for having the same magnitude as that in equation (14.B) for a backward-looking debt, but with sign reversals for both the numerator and denominator.[9]

Lemma Two: There is a unique equilibrium for growth and for public debt, if the debt dynamics are forward-looking.

Substituting (14.F1) into (9) gives the implicit function,

$$F \equiv G - (1-\tau)G^* + \frac{AG}{1+(1-\tau)r - G}(\tau - \gamma) = 0,$$

$$\frac{\partial(\tau-\gamma)}{\partial G} = -\frac{1 + \left(A/\left(1+(1-\tau)r - G\right)^2\right)\left(1+(1-\tau)r\right)(\tau-\gamma)}{AG/\left(1+(1-\tau)r - G\right)} < 0. \qquad (17.F1)$$

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[8] Should these happen to fall beneath their notional value, then bond-holders are effectively receiving what is available from the public finances rather than what they are due. This case might be described as one of "haircuts", if a short-run occurrence; but it more difficult to imagine this possibility in a steady state.

[9] The dynamics are notably different because the size of the debt is determined instantaneously by beliefs of the future well before the capital stock has been given time to adjust. The model outside the steady state will thus demonstrate saddle-path properties with jumps in debt followed by responses in growth that are distributed over time.

The derivative indicates that growth is decreasing in the primary surplus, as $1+(1-\tau)r > G$ and $\tau > \gamma$. Growth is also negatively related to the debt ratio, as

$$\frac{\partial \delta}{\partial(\tau-\gamma)} = \frac{G}{\left(1+(1-\tau)r\right)-G}\left(1+\frac{A\left(1+(1-\tau)\alpha R\right)(\tau-\gamma)}{\left(1+(1-\tau)r-G\right)^2}\right)^{-1} > 0$$

Proposition Two: In the 2-OLG model with forward-looking public debt, there is a degenerate steady state maximum ($G=1$) at $\delta^{\max} = (\tau^{\max}-\gamma)\big/(1-\tau^{\max})r$, where

$$\tau^{\max} = 1+\frac{1}{2G^*}\left(\frac{A}{r}-1\right)-\sqrt{\frac{1}{4G^{*2}}\left(\frac{A}{r}-1\right)^2+\frac{1}{G^*}\frac{A}{r}\left(1-\gamma\right)}$$

The monotonically negative relationship between the debt ratio and economic growth implies that in the steady state the former is highest where the latter is at its steady state minimum of $G=1$. This implies $\delta^{\max} = (\tau^{\max}-\gamma)\big/(1-\tau^{\max})r$ according to equation (14.F) and $1=(1-\tau)G^* - A\delta$, according to equation (9.1). These are solved simultaneously to give the maximum tax rate as above.

There is a bifurcation maximum for the backward-looking debt, but a degenerate maximum for the forward-looking one. This is because the two distinct dynamic cases have different stability conditions, which imply opposite signs for the partial response of public debt to economic growth, $\partial\delta/\partial G$. The crowding-out effect of debt on growth, $\partial G/\partial\delta < 0$, being common to both, thus causes a non-monotonic relationship, a source of a bifurcation, in the first case, but a monotonically decreasing one leading to a degeneracy in the second one.

3.2 A valuation of the steady state maximum when debt is forward-looking

Equations (2) and (9.1) give $R_K = \left(\alpha/(1-\alpha)\right)\left((1+\lambda)/\lambda\right)G^*$. If we fix the debt/tax-free growth factor at is previous level, we may consider various possibilities of the interest rate in this financial efficiency by varying the value of α. We now consider higher annualised returns on capital, r_K^{pa} of 3%, 4% and 5%, and find that setting $\lambda = 0.8$ allows a more set of plausible vales for capital income share of 0.34621, 0.42851 and 0.51414. The debt is increasing in the primary surpluses, so will be highest where public expenditure is minimized at $\gamma = 0$. The results are presented as follows.

Table Two: Values at the degeneracy of the maximum for forward-looking debt with two overlapping generations: $g^{pa*} = 2.5\%$			
	$r^{pa} = 3\%$	$r^{pa} = 4\%$	$r^{pa} = 5\%$
g	0	0	0
τ	18.30%	22.36%	25.61%
δ^{2OLG}	11.80%	9.28%	7.19%
δ^{pa}	33.19%	36.61%	39.09%

We cannot sensibly make a quantitative comparison of the two cases, since they are predicated on the different parameter values underlying the separate stability conditions. In qualitative terms, if the dynamics are forward-looking case, the debt is potentially larger, because the responsive fall in growth is potentially further to a corner point of degeneracy rather than to an interior point of bifurcation, but this effect is countered by the fact that the new maximum in this case is necessarily supported by high taxes, which are have contractionary debt-servicing effects. However, an unambiguous conclusion that may be drawn is that that these two qualitatively different cases each deliver debt maxima that are far below what might be envisaged empirically.

5. A 3-OLG model

5.1 Modifications

This prompts us to generalise the model by dispensing with the standard assumption that each household saves only in a single period of life. This requires that the utility function in equation (5.1) is extended to three periods,

$$U_t = \ln c_t^Y + \lambda \ln c_{t+1}^M + \lambda^2 \ln c_{t+1}^O, \tag{5.2}$$

for consumption when young (*Y*), middle-aged (*M*) and old (*O*); there is also geometric discounting. The household works only in the first two periods, receives a pre-tax wage in each, w_t^Y and w_t^M , and faces the post-tax budget constraint,

$$(1-\tau)w_t^Y + \frac{1}{R^N}(1-\tau)w_t^M = c_t^Y + \frac{1}{R^N}c_{t+1}^M + \frac{1}{{R^N}^2}c_{t+2}^O \text{, where } R^N \equiv 1+(1-\tau)r \text{ .}$$

5.2 The asset flow equations

The two asset flow equations are

$$\begin{aligned} k_{t+1} - (1-\delta)k_t &= (A_{K,t}^Y - 0) + (A_{K,t}^M - A_{K,t-1}^Y) \\ b_t - b_{t-1} &= (A_{B,t}^Y - 0) + (A_{B,t}^M - A_{B,t-1}^Y) \end{aligned} \tag{21}$$

The left-hand-sides represent the *flow supplies*, respectively, of capital issued by firms – as equities, loans, etc – and of public debt issued by the government – as bonds and treasury bills, where an the rate of depreciation of the former is made explicit by the term δ. The right-hand-sides represent the corresponding *flow demands* from young and middle-aged households. The flow demands of the young are synonymous with their stock demands, $A_{K,t}^Y$ and $A_{B,t}^Y$, since the assumption is that they arrive on the scene without having acquired or inherited stocks of assets. Those of the middle-aged are constituted by their current stock demands less the stocks they previously acquired when young, $A_{K,t}^M - A_{K,t-1}^Y$ and $A_{B,t}^M - A_{B,t-1}^Y$. The old, as before, are assumed merely to disburse their entire asset holdings for a final period of consumption without leaving a bequest to their children or, as the model now permits, to their grand-children.

Aggregating equations (21), where $A_t^Y = A_{K,t}^Y + A_{B,t}^Y$ and $A_t^M = A_{K,t}^M + A_{B,t}^M$, gives

$$k_{t+1} - (1-\delta)k_t = A_t^Y + A_t^M - A_{t-1}^Y - (d_t - d_{t-1}) \tag{22}$$

This shows that public debt *flows*, $d_t - d_{t-1}$,crowd-out investment flows, $k_{t+1} - (1-\delta)k_t$. Maintaining the earlier 100% depreciation assumption, $\delta = 1$, as well for these shorter time-periods gives

$$k_{t+1} = A_t^Y + A_t^M - A_{t-1}^Y - (d_t - d_{t-1}). \tag{23}$$

This is to be compared with $k_{t+1} = A_t^Y - d_t$ in earlier equation (3), where $A_t^Y = s_t$, but where, moreover, $d_{t-1} = 0$, so that d_t, *de facto* , is both a stock and a flow.

Certain sign restrictions are necessary for a tenable aggregation. The asset stocks of public debt are strictly positive, $A_{B,t}^Y > 0$, $A_{B,t}^M > 0$, because the government is deemed only to borrow from households. Net flows of capital are also positive, because firms are assumed not to borrow from the government. For present purposes, public debt and capital are perfect substitutes.

However, in the expectation of an income when middle-aged and depending on their preferences, the young might choose to borrow. If so, it is assumed they will borrow on the same terms as firms and so at a common interest rate, $R^L = R^K = \alpha A$: that is, there is no price discrimination – nor differential uncertainty in this extended model.

Financial frictions would lead to lower saving rate so interest, $R^S < R^L$, as well as to the possibility that young households would choose neither to save nor to borrow. In this event, little is gained analytically, because the model simple reverts to the earlier form where each generation saves only in a single period. The model might then be generalized by progressively increasing the number of generations until the periods are sufficiently short to facilitate saving over two periods, but for an analytical rather than an empirical model it surely makes more sense to impose the parameter conditions under which the young as well as the middle-aged will choose to save.

The details are given in an *Appendix*, while the solution is as follows. First, the condition that also the young save is given by

$$\lambda + \lambda^2 > G R_N{}^{-1} \tag{24}$$

Then the solution for economic growth is given by

$$G_{t+1} = B_0 + B_1 G_t{}^{-1} - B_2\left(\delta_t - G_t{}^{-1}\delta_{t-1}\right),$$

$$B_0 \equiv \frac{(1-\tau)(1-\alpha)A\left(\lambda + 2\lambda^2 + R_N{}^{-1}\right)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > 0$$

$$B_1 \equiv \frac{(1-\tau)(1-\alpha)A\left(\lambda^2(R_N-1)-\lambda\right)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > (<) 0 \quad \text{if} \quad \lambda(R_N-1) > (<) 0$$

$$B_2 \equiv \frac{(1+\lambda+\lambda^2)A}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N{}^{-1}} > 0 \tag{9.2}$$

The special case without either debt or taxes is

$$G_{t+1}^* = B_0^* + B_1^* G_t^{*-1},$$

$$B_0^* \equiv \frac{(1-\alpha)A\left(\lambda + 2\lambda^2 + R_N{}^{-1}\right)}{1+\lambda+\lambda^2+(1-\alpha)AR_N{}^{-1}} > 0,$$

$$B_1^* \equiv \frac{(1-\alpha)A\left(\lambda^2(R_N-1)-\lambda\right)}{1+\lambda+\lambda^2+(1-\alpha)AR_N{}^{-1}} > (<) 0, \quad \text{if} \quad \lambda(R_N-1) > (<) 1,$$

$$B_2 \equiv \frac{(1+\lambda+\lambda^2)A}{1+\lambda+\lambda^2+(1-\alpha)AR_N{}^{-1}} > 0, \tag{9.2*}$$

which gives a unique (positively valued) steady-state

$$G^* = B_0^*/2 + \sqrt{(B_0^*/2)^2 + B_1^*} \quad \text{if} \quad B_0^{*2} + 4B_1^* > 0$$

6. Backward-looking public debt in the 3-OLG model

6.1 The condition for two generations to save

First, a backward-looking debt is less favourable to the case of two generations saving, because the necessary backward-stability condition, $G > 1 + (1-\tau)r$, may conflict with the condition that households save also in the first period of their lives, $(\lambda + \lambda^2)(1+(1-\tau)r) > G$. Thus, a judicious choice of parameter values is required in order to incorporate both these features, which can be both be regarded as plausible, if not standard.[10]

6.2 Steady state

Lemma Three: There are two steady states for growth and for public debt, if the debt dynamics are backward-looking with two generations saving, if $(1-\tau)r > 0$.

Substituting (14.B) into (9.2) gives the implicit function,

$$F = G - B_0 - B_1 G^{-1} + B_2 \frac{G-1}{G - \left(1 + (1-\tau)r\right)}(\gamma - \tau),$$

$$\frac{\partial(\gamma-\tau)}{\partial G} = -\left(\frac{1 + B_1 G^2 - B_2(\gamma-\tau) \big/ \left(G - \left(1+(1-\tau)r\right)\right)^2 (1-\tau)r}{B_2\,(G-1) \big/ \left(G - \left(1+(1-\tau)r\right)\right)}\right) \qquad (17.B2)$$

Note that the numerator changes sign *only* if $(1-\tau)r > 0$. Also note that

$$\frac{\partial \delta}{\partial(\gamma - \tau)} = \frac{\left((1 - B_1 G^{-2})(G-R)^2\right)G + B_2(G-1)\gamma}{\left(1 - B_1 G^{-2}\right)\left(G-R\right)^2 - B_2(R-1)\gamma} > 0$$

[10] In the absence of price discrimination, where young households could borrow at the same unsubsidised factor as firms, R_K, the condition for wanting so to do is $(\lambda + \lambda^2)R_K < G$; and thus the condition $(\lambda+\lambda^2)(1+(1-\tau)r) < G < (\lambda+\lambda^2)R_K$ implies a desire to neither save nor borrow.

Proposition Three: In the 3-OLG model with backward-looking public debt, (i) if $r > 0$, there is a maximum public debt at a bifurcation; (ii) if $r \le 0$, there is a degeneracy, where there is no maximum for the public debt.

Part (i) is merely an extension of *Proposition One*. The difference is that the bifurcation will occur at a lower value of the growth rate. Part (ii), however, states that there will be a degeneracy also in the backward-looking case if $r \le 0$, but, the question arises whether this could be regarded as a steady state? If, hypothetically, there is a degeneracy at $G = 1$, if $\tau = 0$, the debt-ratio would be unbounded in the steady state. However, this limiting case indicates the importance of low interest rates for the size of the debt-GDP ratio. This point is demonstrated with various real interest rate values in the next *Table*.

6.3 Numerical values

We maintain the same values for the annual rates of the debt/tax-free growth rate and the return on saving as for the previous case of backward-looking debt with two overlapping generations.

Table Three: Values at the bifurcation of the maximum for a backward-looking debt in the 3-OLG model where two generations save $g^{pa*} = 2.5\%$			
	$r^{pa} = 0.25\%$	$r^{pa} = 1.25\%$	$r^{pa} = 2.25\%$
g^{pa}	0.82%	1.78%	2.37%
γ	27.21%	5.35%	0.17%
δ^{3OLG}	212.15%	45.12%	5.97%
δ^{pa}	374.36%	79.62%	10.53%

There is extreme variation in response to changes in the interest rate. If the difference between the interest rate and the growth rate is small, there is virtually no change from the previous specification of two overlapping generations. However, if there is severe financial repression leading to very low household interest rates, the debt-GDP ratio in the steady state may potentially be very high. There are two reasons for this. First, in the limit where the interest rate converges to zero, the bifurcation converges to the degeneracy of a zero rate of economic growth, which implies negligible changes in the stock of debt – or in debt flows – and, hence, an imperceptible degree of crowding-out. Thus, a low rate of economic growth becomes a cause as well as a consequence of a large public debt. Secondly, a large backward-looking debt, as the accumulation of past primary deficits, is consistent with very low taxes and, hence, a minimal degree of crowding-out through the factor of debt-servicing.

7. Forward-looking public debt in the 3-OLG model

7.1 The steady state

Lemma Four: There is a unique steady state for growth and for public debt, if the debt dynamics are forward-looking with two generations saving.

Substituting (14.F) into (9.2) gives the implicit function,

$$F = G - B_0 - B_1 G^{-1} + B_2 \frac{G-1}{\left(1+(1-\tau)r\right)-G}(\tau-\gamma),$$

$$\frac{\partial(\tau-\gamma)}{\partial G} = -\left(\frac{1+B_1 G^2 + B_2(\tau-\gamma)\big/\left(\left(1+(1-\tau)r\right)-G\right)^2(1-\tau)r}{B_2\,(G-1)\big/\left(\left(1+(1-\tau)r\right)-G\right)}\right) < 0 \qquad \text{(17F2)}$$

Proposition Four: In the 3-OLG model, if public debt is forward-looking, its maximum level is at a degeneracy ($G=1$) where $\delta^{\max} = (\tau-\gamma)/(1-\tau)r$ and it is clearly highest for this previous case where $\gamma = 0$ and $\tau = \tau^{\max}$, and where $\tau = \tau^{\max}$ satisfies $1 - B_0 - B_1 = 0$ from equation (9.2).

This replicates the previous case, but a degeneracy with two generations saving precludes the asset crowding-out effect, leading to potentially higher values.

7.2 Numerical values

We choose the same values for the interest rate as for the 2-OLG version of the model in *Section 4*. $\lambda = 0.6$ is chosen to obtain plausible values for α.

Table Four: Values at the degeneracy for the maximum forward-looking public debt ratio in a 3-OLG model where two generations save			
	$r^{pa} = 3\%$	$r^{pa} = 4\%$	$r^{pa} = 5\%$
g^{pa}	0%	0%	0%
τ	30.71%	32.98%	32.88%
δ^{3OLG}	42.92%	31.47%	22.02%
δ^{pa}	84.71%	77.76%	67.63%

We see that the values, in comparison with those of *Table Two*, are of a higher and more plausible order of magnitude. The similarity of relative modest responses to changes in the interest rate value, however, remains. The associated *average* tax rates appear to be quite low, but these could be increased significantly by raising the assumed value for the debt/tax-free growth rate.

8. Summary and conclusions

The question *how large can the public debt get?* is not only of theoretical interest but also of some practical concern. The Diamond OLG model has proved to be remarkably versatile for addressing a whole range of issues in the areas of public finance and of finite-horizon macroeconomics. We have also found that it is also useful for analysing economies with very large public debt, provided that it is assumed that households save over more than a single period of their lives in order to jettison the default feature of stock-flow synonymity. This allows much larger and empirically sensible values for the stock of public debt stock, because large values may coexist with small flows. A minor generalisation to three overlapping generations thus equips the model to deal better with extreme cases of the phenomenon that it was originally formulated to address. In conclusion, we find that a suitable specification allows the size of the public debt to ascend to empirically plausible magnitudes, thus supporting the continued use of the Diamond OLG model for asking this kind of question.

References

Barro, R.J. (1974), "Are government bonds net wealth?" *Journal of Political Economy*, 82, 1095-1117.

Blanchard, O.J. (1985), "Debts, deficits and finite horizons", *Journal of Political Economy*, 93, 2, 223-247.

Bohn, H. (1995), "The sustainability of budget deficits in a stochastic economy", *Journal of Money Credit and Banking*, 27, 1, 257-271.

Bohn, H. (1998), "The behaviour of public debts and deficits", *Quarterly Journal of Economics*, 113, 3, 949-963.

Braeuninger, M. (2005), "The budget deficit, public debt, and endogenous growth", *Journal of Public Economic Theory*, 7, 5, 827-840.

Chalk, N.A. (2000), "The sustainability of bond-financed deficits: an overlapping generations approach", *Journal of Monetary Economics*, 45, 293-328.

Diamond, P.A. (1965), "National debt in a neoclassical growth model", *American Economic Review*, 55, 1126-1150.

Farmer, K. And J. Zotti (2010), "Sustainable government debt in a two-good, two-country overlapping generations model", *International Review of Economics*, 57, 289-316.

Greiner, A., U. Koeller and W. Semmler (2007), "Debt sustauinability in the European Monetary Union: theory and empirical evidence for selected countries", *Oxford Economic Papers*, 59, 194-218.

Lucas, R.E. (1988), "On the dynamics of economic development", *Journal of Monetary Economics*, 22, 1, 3-42

Nielsen, S.B. (1992), "A note on the sustainability of primary budget deficits", *Journal Of Macroeconomics*, 14, 745-754.

Rankin, N. (2012), "Maximum sustainable government debt in the perpetual youth model", published online as *early view* and forthcoming in the *Bulletin of Economic Research*.

Rankin, N. And B. Roffia (2003), "Maximal sustainable government debt in the overlapping generations model", *The Manchester School*, 71, 3, 217-241.

Roberts, M.A. (2013), "Fiscal rules and the maximum sustainable size of the public debt in the Diamond overlapping generations model", *CFCM Working Paper 2013-07*.

Roberts, M.A. (2014), "A non-monotonic relationship between public debt and economic growth: the effect of financial monopsony", *CFCM Working Paper 2014-02*.

Romer, P.M. (1986), "Increasing returns and long-run growth", *Journal of Political Economy*, 94, 1002-1037

Yaari, M.E. (1965), "Uncertain lifetime, life insurance and the theory of the consumer", *Review of Economic Studies*, 32, 2, 137-150.

Appendix: Solution for economic growth for in the 3-OLG model

Defining $R_N \equiv 1+(1-\tau)r$, the planned (and actual) consumption demands are

$$c_t^Y = \frac{(1-\tau)\left(w_t + R_N{}^{-1} w_{t+1}\right)}{1+\lambda+\lambda^2}, \qquad c_{t+1}^M = \lambda R_N c_t^Y, \qquad c_{t+2}^O = (\lambda R_N)^2 c_t^Y \tag{A1}$$

This implies that the young household's asset holding is

$$A_t^Y = (1-\tau)w_t^Y - c_t^Y = \frac{(1-\tau)\left((\lambda+\lambda^2)w_t^Y - R_N{}^{-1} w_{t+1}^M\right)}{1+\lambda+\lambda^2} \tag{A2}$$

The condition that the young save, $A_t^Y > 0$, is $\lambda+\lambda^2 > GR_N{}^{-1}$. Clearly, the forward-looking dynamic case where $GR_N{}^{-1} < 1$ is more conducive to this possibility, while if $GR_N{}^{-1} > 1$ in the backward-looking case, the condition $\lambda+\lambda^2 > G$ will never hold unless $-1/2+\sqrt{1/4+GR_N{}^{-1}} < \lambda \leq 1$, for which it is necessary that $G < 2R_N$.

The asset position of the middle-aged is

$$A_{t+1}^{M} = \frac{\lambda^2(1-\tau)\left(R_N w_t^Y + w_{t+1}^M\right)}{1+\lambda+\lambda^2} \tag{A3}$$

which is strictly positive as there is no earned income when old. Subtracting (A2) from (A3) gives the asset flow demand of the *future* middle-aged as

$$A_{t+1}^{M} - A_t^Y = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_t^Y + \left(\lambda^2 + R_N{}^{-1}\right)w_{t+1}^M\right)}{1+\lambda+\lambda^2}$$

or for the *current* middle-aged as

$$A_t^{M} - A_{t-1}^Y = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_{t-1}^Y + \left(\lambda^2 + R_N{}^{-1}\right)w_t^M\right)}{1+\lambda+\lambda^2} \tag{A4}$$

Where there are no seniority or productivity effects, $w_t^Y = w_t^M = w_t$, adding (A2) and (A4) gives

$$A_t - A_{t-1} = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_{t-1} + \left(\lambda+2\lambda^2 + R_N{}^{-1}\right)w_t - R_N{}^{-1}w_{t+1}\right)}{1+\lambda+\lambda^2} \tag{A5}$$

Equation (20) then implies

$$k_{t+1} = \frac{(1-\tau)\left(\left(\lambda^2(R_N-1)-\lambda\right)w_{t-1} + \left(\lambda+2\lambda^2 + R_N{}^{-1}\right)w_t - R_N{}^{-1}w_{t+1}\right)}{1+\lambda+\lambda^2} - (d_t - d_{t-1})$$

which with (2), after rearranging gives

$$k_{t+1} = \frac{(1-\tau)(1-\alpha)A\left(\left(\lambda^2(R_N-1)-\lambda\right)k_{t-1} + \left(\lambda+2\lambda^2 + R_N{}^{-1}\right)k_t\right) - (1+\lambda+\lambda^2)\left(d_t - d_{t-1}\right)}{1+\lambda+\lambda^2 + (1-\tau)(1-\alpha)AR_N{}^{-1}}$$

and, using the definitions in (8)

$$G_{t+1} = \frac{(1-\tau)(1-\alpha)A\left(\left(\lambda^2(R_N-1)-\lambda\right)G_t{}^{-1} + \left(\lambda+2\lambda^2 + R_N{}^{-1}\right)\right) - (1+\lambda+\lambda^2)A\left(\delta_t - G_t{}^{-1}\delta_{t-1}\right)}{1+\lambda+\lambda^2 + (1-\tau)(1-\alpha)AR_N{}^{-1}}$$

$$G_{t+1} = B_0 + B_1 G_t^{-1} - B_2\left(\delta_t - G_{t-1}\delta_{t-1}\right),$$

$$B_0 = \frac{(1-\tau)(1-\alpha)A\left(\lambda + 2\lambda^2 + R_N^{-1}\right)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N^{-1}} > 0$$

$$B_1 = \frac{(1-\tau)(1-\alpha)A\left(\lambda^2(R_N-1)-\lambda\right)}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N^{-1}} > (<)0 \quad \text{if} \quad \lambda(R_N-1) > (<)0$$

$$B_2 = \frac{(1+\lambda+\lambda^2)A}{1+\lambda+\lambda^2+(1-\tau)(1-\alpha)AR_N^{-1}} > 0$$