The University of Manchester
Brooks World Poverty Institute

MANCHESTER 1824

Proposing indicators to measure achievement and shortfall inequality consistently

Casilda Lasso de la Vega[1]

Oihana Aristondo[2]

[1] University of the Basque Country, Bilbao, Spain

casilda.lassodelavega@ehu.es

[2] University of the Basque Country, Barakaldo, Spain

oihana.aristondo@ehu.es

May 2010

Brooks World Poverty Institute
ISBN : 978-1-907247-19-4

BWPI Working Paper 120

Creating and sharing knowledge to help end poverty

www.manchester.ac.uk/bwpi

Abstract

In several economic fields, such as those related to health or education, the individuals' characteristics are measured by bounded variables. Accordingly, these characteristics may be indistinctly represented by achievements or shortfalls. A difficulty arises when inequality needs to be assessed. One may focus either on achievements or on shortfalls, but the respective inequality rankings may lead to contradictory results. In this note, we propose a procedure to define indicators to measure consistently the achievement and shortfall inequality. Specifically, we derive measures which are invariant under ratio-scale or translation transformations, and a decomposable measure also is proposed.

Keywords: Inequality measurement, Health inequality, Achievements, Shortfalls

Casilda Lasso de la Vega is at the Department of Applied Economics IV, University of the Basque Country, Bilbao, Spain.

Oihana Aristondo is at the Department of Applied Mathematics, University of the Basque Country, Barakaldo, Spain.

Acknowledgements: This research has been partially supported by the Spanish Ministry of Education and Science under project SEJ2009-11213, cofunded by FEDER and by the Basque government under the project GIC07/146-IT-377-07. The first author gratefully acknowledges the hospitality of the Brooks World Poverty Institute (BWPI) of the University of Manchester, where she was visiting, supported by the Basque government under the programme MV-2010-1-20, when the fist version of this paper was prepared.

We thank our colleague Ana Urrutia for her useful comments and suggestions.

.

1. Introduction

A number of recent papers have highlighted the difficulties in measuring inequality of a distribution that can be described either in terms of achievements or shortfalls (among them Clarke et al., 2002, Erreygers, 2009, and Lambert and Zheng, 2010). This situation arises in different economic fields in which bounded variables are involved, particularly in the measurement of health inequality. As stressed in the papers mentioned above, the choice between achievement and shortfall inequality measurement is not innocuous, since different choices may lead to contradictory results.

Erreygers (2009), from now on Erreygers, characterises two indicators, both depending on the distribution bounds, which measure achievement and shortfall inequality identically. The square of one of them is decomposable, in the sense that the overall inequality can be decomposed as the sum of the inequality between-groups and the inequality within-groups (Shorrocks, 1980). In turn, Lambert and Zheng (2010), henceforth Lambert-Zheng, introduce a weaker property to measure achievement and shortfall inequality consistently, and show that all relative and intermediate inequality indices fail their requirement. They also identify two classes of absolute inequality indices, according to which the measure of achievement and shortfall inequality is identical, and they show that the variance is the only subgroup decomposable measure. They also prove that no consistent index exits that is sensitive to transfers at different parts of the distribution.

A difference between these two approaches should be noted. Whereas Lambert-Zheng's setting concentrates on the standard inequality measures, the indicators proposed by Erreygers, as already mentioned, depend on the distribution bounds. The implications of this difference will be discussed.

In connection with this, we propose a simple procedure that allows us to transform any inequality measure into an indicator which measures the achievement and the shortfall inequality equally. Similarly to Erreygers, our proposal depends on the maximum level of achievements. We show that some of the properties enjoyed by the original index are inherited by its transformation. Accordingly, measures able to capture, both ratio-scale and translation invariant, inequality may be obtained with our procedure and a decomposable index is also identified. Decomposability is quite useful in applied analysis, since it allows policy makers to identify the sources of inequality and to target them, in order to achieve a maximum reduction in inequality levels.

The measure of inequality depends deeply on the selected indicator. The possibility to assess inequality with several indices may add robustness to the results. We think this is the basic contribution of our work.

This note is organised as follows. In Section 2 the basic notations and definitions are presented. Section 3 introduces the procedure to derive indicators for which the achievement inequality and the shortfall inequality are equal, and some implications of the method are discussed.

2. Notation and basic definitions

We consider a population consisting of $n \geq 2$ individuals. An *achievement distribution* is represented by a vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbf{R}^n_{++}$, where x_i represents individual *i*'s achievement. We denote by α the maximum level of achievements. We assume that α is given and fixed. Thus, $s_i = \alpha - x_i$ represents individual *i*'s shortfall and $\mathbf{s} = (s_1, s_2, ..., s_n) \in \mathbf{R}^n_{++}$ is the *shortfall distribution* associated with **x**.

We let $D = \bigcup_{n \geq 2} \mathbf{R}^n_{++}$ represent the set of all finite dimensional distributions, and $\mu_{\mathbf{y}}$ and $n_{\mathbf{y}}$ stand, respectively, for the mean and population size for any $\mathbf{y} \in D$. We use the notation $\mathbf{1} = (1, ..., 1)$ and $\lambda\mathbf{1} = (\lambda, ..., \lambda)$. Given two distributions $\mathbf{y}, \mathbf{y}' \in D$, we say that $\mathbf{y}'$ is obtained from $\mathbf{y}$ by a *progressive transfer* if there exist two individuals $i, j \in \{1, ..., n\}$ and $h > 0$ such that $y_i' = y_i + h \leq y_j - h = y_j'$ and $y_k' = y_k$ for every $k \in \{1, ..., n\} \setminus \{i, j\}$.

In the standard income literature, an inequality index *I* is a real valued function $I : D \rightarrow \mathbf{R}$ which fulfils the following properties.

Pigou-Dalton transfer principle (TP). $I(\mathbf{y}') < I(\mathbf{y})$ whenever $\mathbf{y}'$ is obtained from $\mathbf{y}$ by a progressive transfer.
Normalisation (NOR). $I(\lambda\mathbf{1}) = 0$ for all $\lambda > 0$.
Symmetry (SYM). $I(\mathbf{y}) = I(\mathbf{y}')$ whenever $\mathbf{y} = \Pi\mathbf{y}'$ for some permutation matrix Π.
Replication invariance (RI). $I(\mathbf{y}) = I(\mathbf{y}')$ whenever $\mathbf{y}' = (\mathbf{y}, \mathbf{y}, ..., \mathbf{y})$ with $n_{\mathbf{y}'} = mn_{\mathbf{y}}$ for some positive integer *m*.

The crucial axiom in inequality measurement is the *Pigou-Dalton transfer principle*, which requires that a transfer form a richer person to a poorer one decreases inequality. In this field, it is usually assumed that the indices are *normalised* with the inequality level equal to 0 when everybody has exactly the same distribution value. *Symmetry* establishes that the inequality index should be insensitive to a reordering of the individuals. Finally, *replication invariance* allows populations of different sizes to be compared. These four properties are considered to be inherent to the concept of inequality and have come to be accepted as basic properties for an inequality index.

3. The consistent indicator *I**

3.1 The procedure

In this paper we are concerned with indicators able to measure the achievement and the shortfall inequality consistently. The following proposition provides a procedure to derive measures for which these two inequality levels are equal, that is, they are *perfect complementary*, according to Erreygers's designation.[1]

First of all, given an inequality index *I*, we define the *consistent index associated with I*, which will be denoted by *I**, as the measure, that for each distribution **y**, with α as the maximum achievement level, takes the following value:

$$I^*(\mathbf{y}) = \frac{I(\mathbf{y}) + I(\alpha\mathbf{1} - \mathbf{y})}{2}$$

Note that **y** may represent indistinctly an achievement or a shortfall distribution.

Although the index *I** depends on α, for the sake of simplicity, it is omitted in the notation.

Proposition 1*. The consistent index I* associated with an inequality measure I satisfies TP, NOR, SYM and RI. It also holds that* $I^*(\mathbf{x}) = I^*(\mathbf{s})$*, where **x** and **s** are the achievement and the associated shortfall distributions, respectively.*

Proof. It is clear that *I** satisfies NOR, SYM and RI as *I* does. To prove that *I** also fulfils TP, let us assume that $\mathbf{x'}$ is derived from $\mathbf{x}$ by a progressive transfer. Then, $\mathbf{s'} = \alpha\mathbf{1} - \mathbf{x'}$ is also derived from $\mathbf{s} = \alpha\mathbf{1} - \mathbf{x}$ by a progressive transfer. In fact, a progressive transfer among two individuals' achievements leads to an increment in the richer person's shortfall, whereas the poorer person's shortfall decreases. Since the richer person's shortfall is smaller than the poorer one's, a progressive transfer of achievements is equivalent to a progressive transfer of shortfalls. Consequently, under a progressive transfer both $I(\mathbf{x})$ and $I(\mathbf{s})$ are bound to decrease and so does I^*. Q.E.D.

*I** is not properly a standard inequality measure, since it depends on α. Nevertheless, it fulfils all the properties usually assumed for an inequality index, mainly TP. So it is able to capture the distribution inequality. In fact, *I** is the same kind of indicator as those proposed by Erreygers.

As $I^*(\mathbf{x}) = I^*(\mathbf{s})$, proposition 1 allows us to transform any inequality measure into a *perfect complementary* indicator. If *I* is already *perfect complementary*, then $I^*(\mathbf{y}) = I(\mathbf{y})$. Thus, all the *perfect complementary* measures proposed up to now appear as specific cases in this setting.

[1] This is a stronger condition than Lambert-Zheng's consistency condition, which only demands that the inequality rankings of the achievements and the shortfalls coincide.

In addition, proposition 1 opens up a wide range of possibilities to derive *perfect complementary* indices, both rank-dependent and rank-independent. There are some consistent indices which are particularly easy to compute.

On the one hand, we derive the consistent indicator associated with the Gini coefficient. Since $\mathrm{G}(\mathbf{s}) = (\mu_x/\mu_s)\mathrm{G}(\mathbf{x})$, then $\mathrm{G}^*(\mathbf{y}) = \mathrm{G}(\mathbf{y})\left[\alpha/2(\alpha-\mu_y)\right]$. A similar expression holds for the coefficient of variation. As $\mathrm{CV}(\mathbf{s}) = (\mu_x/\mu_s)\mathrm{CV}(\mathbf{x})$, then $\mathrm{CV}^*(\mathbf{y}) = \mathrm{CV}(\mathbf{y})\left[\alpha/2(\alpha-\mu_y)\right]$.

On the other hand, we get the first Theil measure (Theil, 1967) defined by $T_0(\mathbf{y}) = \sum_{1\le i\le n} \log(\mu/y_i)/n$. Its consistent indicator, which will play a role in this paper, is computed as $T_0^*(\mathbf{x}) = \sum_{1\le i\le n} \log(\mu_x \mu_s / x_i s_i)/2n$.

In what follows, we examine whether some additional properties fulfilled by *I* are inherited by *I**.

3.2 Invariance conditions

Let us begin with the invariance conditions. There are two often invoked in the literature. The first one requires that the inequality level remains unchanged under proportional changes in all the values.

Ratio-scale invariance. $I(\lambda\mathbf{y}) = I(\mathbf{y})$ for all $\lambda > 0$.

When ratio-scale variables are involved, this property guarantees that the inequality level is insensitive to variations in the unit of measurement of the variables.

The second invariance condition demands that the same increase in all the distribution values does not change the inequality level. This property is formalised as follows.

Translation invariance. $I(\mathbf{y}+\eta\mathbf{1}) = I(\mathbf{y})$ for all η whenever $\mathbf{y}+\eta\mathbf{1} \in D$.

Absolute indices are those that are translation invariant.

These two invariance conditions may be encompassed into a more general invariance property that requires that the inequality level does not change under linear transformations. If achievements are measured with cardinal variables, this principle implies that the inequality levels are independent from the units in which the achievements are measured.

Cardinal invariance. $I(\lambda\mathbf{y}+\eta\mathbf{1}) = I(\mathbf{y})$ for all $\lambda > 0$ whenever $\lambda\mathbf{y}+\eta\mathbf{1} \in D$.

These three conditions identify some kind of transformations which do not affect the inequality level. In this paper we assume, similarly to Erreygers, that when an achievement distribution is modified under some kind of transformation, the maximum achievement level will be affected by the same transformation. Consequently, the shortfall distribution will be transformed in the same way. The main implication of this assumption is that a proportional increase in all the achievements will increase the shortfall distribution in the same proportion. This is the difference between the ratio-scale invariant indices considered in this note and the relative ones in Lambert-Zheng's paper. For the latter, the maximum achievement level behaves as a fixed threshold. Accordingly, any proportional increment in the individual achievements implies a decrease in the shortfalls. However, under their setting, the shortfall inequality rankings may be reversed with changes in the units in which variables are measured. In other words, the unit-consistency property, in the sense proposed by Zheng (2007), may not be fulfilled.

Nevertheless, this difference does not exist with respect to the translation invariance property. In fact, note that if *I* is translation invariant, then $\mathrm{I}(\alpha\mathbf{1}\text{-}\mathbf{x})=\mathrm{I}((\alpha+\eta)\mathbf{1}\text{-}(\mathbf{x}+\eta\mathbf{1}))=\mathrm{I}(\alpha\mathbf{1}\text{-}(\mathbf{x}+\eta\mathbf{1}))$. Thus, the translation invariant indices, according to this paper's definition, are the same as the absolute indices invoked by Lambert-Zheng.

The following proposition shows that these invariance conditions are inherited by the associated consistent indices.

Proposition 2. *Let $I : D \to \mathbf{R}$ be an inequality index and the consistent indicator I*.*

(i) If I is ratio-scale invariant, then I is also ratio-scale invariant.*
(ii) If I is translation invariant, then I is also translation invariant.*
(iii) If I is cardinal invariant, then I is also cardinal invariant.*

Proof. It is straightforward to take into account that if the individuals' achievements are modified in the same proportion, then individual *i*'s shortfall becomes $\lambda s_i = \lambda\alpha - \lambda x_i$. Similarly, this reasoning holds for (iii).

Proposition 2 shows that the invariance conditions fulfilled by the associated consistent index depend on the conditions satisfied by the original measure. Specifically, all the relative measures, such as the Gini-coefficient, the S-Gini family (Donaldson and Weymark, 1980), the coefficient of variation, the Generalised Entropy family (Shorrocks, 1980) or the Atkinson family (Atkinson, 1970) generate ratio-scale invariant measures, insensitive to changes in the measurement unit, and so unit-free, which are *perfect complementary*.

A similar conclusion holds for the absolute measures. Since our absolute indicators coincide with Lambert-Zheng's, this case may be of special interest, because all the results derived by them for the consistent absolute indices may be applied to the absolute consistent indicators *I**.

3.3 Sensitivity conditions

Transfer sensitivity conditions (Kolm, 1976 and Shorrocks and Foster, 1987) demand that the inequality measure is more sensitive to transfers lower down the distribution. Lambert-Zheng establishes that no consistent inequality measure exists that satisfies the transfer sensitivity axiom. The same simple example they introduce to prove this result may be used in our setting. Hence no consistent indicator associated with any inequality measure fulfils the transfer sensitive axiom.

3.4 Decomposability

In many applied analyses, the population is split into groups according to social characteristics such as region, race, gender, and so on. In these cases, it is quite useful to invoke properties which allow the inequality in each group to be related to overall inequality. An often used requirement proposed by Shorrocks (1980) is to demand that the overall inequality may be decomposed as the sum of the between- and the within- group components. The between-group component is defined as the inequality level of a hypothetical distribution, in which each person's distribution values are replaced by the mean of their subgroup. The within-group component is a weighted sum of the subgroup inequality levels.

If this axiom is fulfilled, it is possible not only to identify subgroups where inequality is particularly high, but also to evaluate their contribution to overall inequality. Thus it is quite useful in applied analysis, since it allows policy makers to target these groups in order to achieve a maximum reduction in inequality levels.

One implication of this property is the subgroup consistency property (Shorrocks, 1984), which requires that if inequality in one group increases, overall inequality should also increase. Both Erreygers and Lambert-Zheng seek decomposable consistent indices in their respective frameworks. Whereas the Gini-type index characterised by the former is not decomposable – in fact it is not even subgroup consistent – the square of the second satisfies decomposability. In turn, Lambert-Zheng shows that the only consistent inequality index which is decomposable is the variance. We investigate whether any of our consistent indicators are decomposable.

To formalise this decomposition assumption, suppose that the population of n individuals is split into $\mathrm{J} \geq 2$ mutually exclusive subgroups, with distribution $\mathbf{y}^{\mathrm{j}} = \left(\mathrm{y}_1^{\mathrm{j}},...,\mathrm{y}_{\mathrm{n_j}}^{\mathrm{j}}\right)$, means $\mu_{\mathrm{j}} = \mu_{\mathbf{y}^{\mathrm{j}}}$ and population sizes $\mathrm{n_j} = \mathrm{n}_{\mathbf{y}^{\mathrm{j}}}$ for all $\mathrm{j}=1,...,\mathrm{n}$. Let inequality in group j be written $\mathrm{I_j} = \mathrm{I}\left(\mathbf{y}^{\mathrm{j}}\right)$. Let us denote by $\mathbf{y}^{\mathrm{B}} = \left(\mu_1 \mathbf{1}_{n_1},...,\mu_{\mathrm{J}} \mathbf{1}_{n_J}\right)$ the distribution in which each person's distribution value is replaced by their subgroup mean.

Decomposability. An index *I* is *decomposable* if the following relationship holds:

$$I\left(\mathbf{y}^{1},...,\mathbf{y}^{1}\right) = I\left(\mathbf{y}^{W}\right) + I\left(\mathbf{y}^{B}\right) = \sum_{\mathbf{j}=1}^{J} w_j I_j + I\left(\mathbf{y}^{B}\right)$$

where w_j is the weight attached to subgroup $j = 1,...,J$.

All the Generalised Entropy measures are decomposable (Shorrocks, 1980). Specifically, the decomposition of the first Theil measure is expressed as follows:[2]

$$\mathrm{T}_0\left(\mathbf{y}^1,...,\mathbf{y}^1\right)=\mathrm{T}_0\left(\mathbf{y}^{\mathrm{W}}\right)+\mathrm{T}_0\left(\mathbf{y}^{\mathrm{B}}\right)=\sum_{\mathbf{j}=1}^{\mathrm{J}}\frac{\mathrm{n}_\mathrm{j}}{\mathrm{n}}\mathrm{T}_{0\,\mathrm{j}}+\mathrm{T}_0\left(\mathbf{y}^{\mathrm{B}}\right)$$

The following proposition shows that the consistent index T_0^* associated with the Theil index is also decomposable.

Proposition 4. *The consistent index associated with the first Theil measure,* T_0^**, is a decomposable measure for which the following decomposition holds:*

$$T_0*\left(\mathbf{y}^1,...,\mathbf{y}^1\right)=T_0*\left(\mathbf{y}^W\right)+T_0*\left(\mathbf{y}^B\right)=\sum_{\mathbf{j}=1}^{J}\frac{n_j}{n}T_{0\,j}*+T_0*\left(\mathbf{y}^B\right)$$

Proof. $T_0*\left(\mathbf{x}^1,...,\mathbf{x}^1\right)=\dfrac{T_0\left(\mathbf{x}^1,...,\mathbf{x}^1\right)+T_0\left(\mathbf{s}^1,...,\mathbf{s}^1\right)}{2}$ by definition

$=\dfrac{1}{2}\left(\sum_{\mathbf{j}=1}^{\mathrm{J}}\dfrac{\mathrm{n}_\mathrm{j}}{\mathrm{n}}\mathrm{T}_{0\mathrm{j}}\left(\mathbf{x}^\mathrm{j}\right)+\mathrm{T}_0\left(\mathbf{x}^\mathrm{B}\right)+\sum_{\mathbf{j}=1}^{\mathrm{J}}\dfrac{\mathrm{n}_\mathrm{j}}{\mathrm{n}}\mathrm{T}_{0\mathrm{j}}\left(\mathbf{s}^\mathrm{j}\right)+\mathrm{T}_0\left(\mathbf{s}^\mathrm{B}\right)\right)$ since T_0 is decomposable

$=\dfrac{1}{2}\left(\sum_{\mathbf{j}=1}^{\mathrm{J}}\dfrac{\mathrm{n}_\mathrm{j}}{\mathrm{n}}\left(\mathrm{T}_{0\mathrm{j}}\left(\mathbf{x}^\mathrm{j}\right)+\mathrm{T}_{0\mathrm{j}}\left(\mathbf{s}^\mathrm{j}\right)\right)+\mathrm{T}_0\left(\mathbf{x}^\mathrm{B}\right)+\mathrm{T}_0\left(\mathbf{s}^\mathrm{B}\right)\right)$ operating

$=T_0*\left(\mathbf{x}^W\right)+T_0*\left(\mathbf{x}^B\right)$ by definition.

Since the weights in the within-group component depend only on the subgroup population shares, this decomposition also satisfies the path independent property proposed by Foster and Shneyerov (2000). Contrary to what happens to most of the decompositions, the variations in between-group inequality as measured by this index do not affect the within-group term. In addition, this decomposition allows policy makers easily to compute the contribution of each group inequality to the overall inequality.

4. Conclusions

This note proposes a procedure to derive indicators capable of measuring achievement inequality and shortfall inequality equally. Among them, either ratio-scale or translation invariant indices may be chosen. A decomposable indicator is also obtained.

Since relatively few consistent indicators exist in the literature, and inequality measurement depends deeply on the indicator chosen, we hope that this note may contribute to the robustness of the results obtained in empirical applications.

[2] In fact, the Theil measure is not only decomposable but it is also the only generalised entropy index which is path independent according to Foster and Shneyerov (2000).

References

Atkinson, A.B. (1970). 'On the measurement of inequality'. *Journal of Economic Theory*, 2, 244-263.

Clarke, P.M., Gerdtham, U.G., Johannesson, M., Bingefors, K., and Smith, L. (2002). 'On the measurement of relative and absolute income-related health inequality'. *Social Science and Medicine*, 55, 1923-1928.

Donaldson, D. and Weymark, J. A. (1980). 'A single parameter generalization of the Gini index and inequality'. *Journal of Economic Theory*, 22, 67-86.

Erreygers, G. (2009). 'Can a single indicator measure both attainment and shortfall inequality?' *Journal of Health Economics*, 28, 885-893.

Foster, J., and Shneyerov, A. (2000). 'Path independent inequality measures'. *Journal of Economic Theory*, 91, 199-222.

Kolm, S. (1976). 'Unequal inequalities'. *International Journal of Economic Theory*, 12, 416-442.

Lambert, P. and Zheng, B. (2010). 'On the consistent measurement of achievement and shortfall inequality'. Working paper series, University of Colorado Denver.

Shorrocks, A. F. (1980). 'The class of additively decomposable iInequality measures'. *Econometrica*, 48, 613-625.

Shorrocks, A. F. (1984). 'Inequality decomposition by population subgroups'. *Econometrica*, 52, 1369-1385.

Shorrocks, A. F., and Foster J. (1987). 'Transfer sensitive inequality measures'. *Review of Economic Studies*, 54, 485-497.

Theil, H. (1967). *Economics and Information Theory*. Amsterdam: North Holland.

Zheng, B. (2007). 'Unit-consistent decomposable inequality measures'. *Economica*, 74, 97-111.

The University of Manchester
Brooks World
Poverty Institute

MANCHESTER
1824

Executive Director
Professor David Hulme

Research Director
Professor Armando Barrientos

Contact:

Brooks World Poverty Institute
The University of Manchester
Humanities Bridgeford Street Building
Oxford Road
Manchester
M13 9PL
United Kingdom

Email: bwpi@manchester.ac.uk

www.manchester.ac.uk/bwpi

The Brooks World Poverty Institute (BWPI) creates and shares knowledge to help end global poverty.

BWPI is multidisciplinary, researching poverty in both the rich and poor worlds.

Our aim is to better understand why people are poor, what keeps them trapped in poverty and how they can be helped - drawing upon the very best international practice in research and policy making.

The Brooks World Poverty Institute is chaired by Nobel Laureate, Professor Joseph E. Stiglitz.

www.manchester.ac.uk/bwpi